Computational Complex Analysis : : Class 30

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Stereographic Projection of a Sphere on a Plane

$N = (0, 0, 1)$ and $p = (p_1, p_2, p_3)$, $z = (x, y, 0)$.

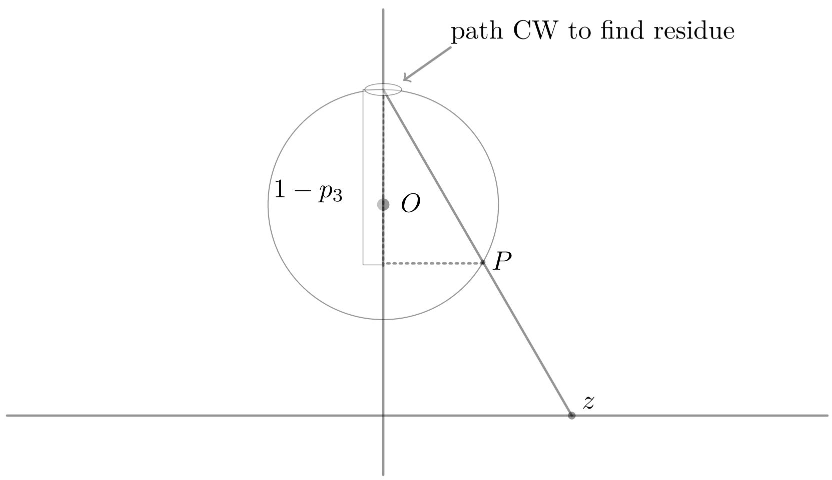

Figure 1: Steregraphic projection (Class 30)

$$x = \frac{p_1}{1 - p_3} \quad y = \frac{p_2}{1 - p_3} \quad z = \frac{p_1 + ip_2}{1 - p_3}$$

$$|z|^2 = \frac{p_1^2 + p_2^2}{(1 - p_3)^2} = \frac{1 - p_3^2}{(1 - p_3)^2} = \frac{1 + p_3}{1 - p_3}$$

Go back to solve for p_3. We say that this sphere is a Riemann Sphere.

Definition 1. If a function $f(z)$ defined for all large $|z|$ as $|z| > R$, then define the behavior of $f(z)$ for $|z| > R$ and $z = \infty$ by saying its the same as the behavior of $f(\frac{1}{z})$ near $z = 0$.
if f is defined for all of z then we say that at infinity f has a removable singularity, if and only if $f(\frac{1}{z})$ has a removable singuality.
We say f has a pole at ∞ if $f(\frac{1}{z})$ has zero at origin.
We say f has essential singularity at ∞ if $f(\frac{1}{z})$ has an essential singularity at the origin.

Definition 2. Residue of f at infinity,

$$\mathrm{Res}\,(f,\infty) = \frac{1}{2\pi i}\int_C f(z)\mathrm{d}z$$

Change of variables, $z \to \frac{1}{z}$ and then,

$$\mathrm{Res}(f,\infty) = \frac{1}{2\pi i}\int_{CW} f(z') - \frac{1}{z^2}\mathrm{d}z$$

$$= \frac{1}{2\pi i}\int_{CCW} f(\frac{1}{z})\mathrm{d}z\,\frac{1}{z^2}$$

$$\mathrm{Res}\left(\frac{f(\frac{1}{z})}{z^2}, 0\right)$$

The Riemann Sphere is the first example of a "complex manifold", this complex manifold is compact. Other examples are there.

Holomorphic function on this manifold is called elliptic function.

Function of $\mathbb{C}$ of the form,

$$f(z) = \frac{az+b}{cz+d}$$

Then $a, b, c, d \in \mathbb{C}$, and $ad - bc \neq 0$. This function is called a Mobius Function with several nice properties related to 2×2 matrices.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} az+bw \\ cz+dw \end{pmatrix}$$

We can say

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} az+b \\ cz+d \end{pmatrix}$$

Extension can be made,

$$f(z) = \frac{az+b}{cz+d} \quad g(z) = \frac{Az+B}{Cz+D}$$

$f \cdot g$ results the matrix,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Now on inverses,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

We are not worried about $ad - bc$ in the second part of the above equation, so,

$$f(z) = \frac{z-i}{2z+3} \to \frac{3z+i}{-2z+1}$$

But now we can think about

$$\frac{az+b}{cz+d}$$

at $z = \infty$. You get,

$$\frac{a}{c}$$

We are going to show, if 3 distinct numbers of $\mathbb{C} \cup \{\infty\}$ are given, and also 3 other distincts are given in order then $\exists 1$ and only 1 mobius function which maps the first 3 onto the second three in order.

Given 3 distinct numbers, $u, v, w \in \mathbb{C}$ there exists 1 Mobius function such that,

$$f(u) = 0 \quad f(v) = \infty \quad f(w) = 1$$

To make this happen,

$$f(z) = \frac{z-u}{z-v}\frac{w-v}{w-u}$$

Also,

$$f(\infty) = 0 \quad f(v) = \infty \quad f(w) = 1$$
$$f(z) = \frac{w-v}{z-v}$$

Also,

$$f(u) = 0 \quad f(\infty) = \infty \quad f(v) = 1$$

$$f(z) = \frac{z-u}{z}\frac{v}{v-u}$$

Also,

$$f(u) = 0 \quad f(v) = \infty \quad f(\infty) = 1$$
$$f(z) = \frac{z-u}{z-v}$$

Now if you want to do this,

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \to \begin{pmatrix} U \\ V \\ W \end{pmatrix}$$

Do this,

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \to (0, \infty, 1) \to \begin{pmatrix} U \\ V \\ W \end{pmatrix}$$

Therefore the Mobius functions form a group with group multiplication being composition. These matrices do not form a group.