Computational Complex Analysis : : Class 29

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Last class we found that,

$$\frac{1}{\Gamma(z)} = \lim_{n\to\infty} \frac{z(z+1)\cdots(z+n)}{n^z n!}$$

Valid for all $z \in \mathbb{C}$. People use the word entire holomorphic function. Slang for holomorphic function defined on the "entire" complex plane. We can rewrite this as an infinite product,

$$\prod_{n=1}^{\infty} \phi_n = \lim_{W\to\infty} \prod_{n=1}^{N} \phi_n$$

$$\frac{z(z+1)\cdots(z+n)}{n^z n!} = \frac{z}{n^z}\frac{z+1}{1}\frac{z+2}{2}\cdots\frac{z+n}{n}$$

The above thing diverges. Weierstrass does this,

$$= z\frac{\left(1+\frac{z}{1}\right)e^{-\frac{z}{1}}\left(1+\frac{z}{2}\right)e^{-\frac{z}{2}}\cdots\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}}{n^z}e^{z\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)}$$

The upper right thing blows up.

$$= z\frac{\left(1+\frac{z}{1}\right)e^{-\frac{z}{1}}\left(1+\frac{z}{2}\right)e^{-\frac{z}{2}}\cdots\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}}{e^{z\ln n}}e^{z\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)}$$

$$= z\left(\left(1+\frac{z}{1}\right)e^{-\frac{z}{1}}\left(1+\frac{z}{2}\right)e^{-\frac{z}{2}}\cdots\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}\right)e^{z\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)}e^{-z\ln n}$$

So what we can write now is that,

$$\frac{z(z+1)(\cdots)(z+n)}{n^z n!} = z\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)e^{-z/k}\times\left(e^{z(1+1/2+\cdots+1/n)-\ln n}\right)$$

The thing goes to limit,

$$1 + 1/2 + \cdots + 1/n - \ln(n) \to \gamma$$

γ is the **Euler-Mascheroni** Number.

There is a limit hence,

$$\boxed{\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)e^{-\frac{z}{k}}}$$

We had in last class,

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin \pi a} \qquad (0 < a < 1)$$

$$\frac{\pi}{\sin \pi z} = \Gamma(z)\Gamma(1-z) = \lim \frac{n^z n!}{z(z+1)\cdots(z+n)} \lim \frac{n^{1-z} n!}{(1-z)(2-z)\cdots(n+1-z)}$$

$$= \lim \frac{n(n!)^2 \quad (-1)^n}{z(z+1)(\cdots)(z+n)\times(z-1)(z-2)(\cdots)(z-n-1)}$$

$$= \lim \frac{n(n!)^2 \quad (-1)^{n+1}}{z(z^2-1)(z^2-4)(\cdots)(z^2-n^2)}\frac{1}{z-n-1}$$

$$= \lim \frac{n(n!)^2 \quad (-1)}{z(1-z^2)(\cdots)(n^2-z^2)} \frac{1}{z-n-1}$$

$$= \lim \frac{1}{z\left(1-\frac{z^2}{1}\right)\left(1-\frac{z^2}{2^2}\right)(\cdots)\left(1-\frac{z^2}{n^2}\right)} \frac{1}{z-n-1}$$

$$\frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)$$