Computational Complex Analysis : : Class 28

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Showing a sine relation

The provisional definition for the Γ function

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \mathrm{d}t$$

This is defined for $\mathrm{Re}(z) > 0$. We have extended Γ function to a holomorphic function defined on,

$$\mathbb{C} \setminus \{-1, -2, \dots\}$$

We rewrite

$$\Gamma(z) = 2 \int_0^\infty t^{2z-1} e^{-t^2} \mathrm{d}t$$

We use this to find,

$$\Gamma(z)\Gamma(w) = \Gamma(z+w) \cdot 2 \int_0^{\pi/2} (\sin\theta)^{2z-1} (\cos\theta)^{2w-1} \mathrm{d}\theta$$

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} \mathrm{d}t$$

$a > 0, b > 0$,

$$= 2\int_0^{\pi/2} (\sin\theta)^{2a-1}(\cos\theta)^{2b-1}\mathrm{d}\theta$$

$$B(a,b) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$$

Brown and Churchill Page 285 shows,

$$\int_0^\infty \frac{x^{-a}}{1+x} \mathrm{d}x = \frac{\pi}{\sin(a\pi)}$$

For $0 < a < 1$.

A new calculation would be for $0 < a < 1$,

$$\Gamma(a)\Gamma(1-a) = B(a, 1-a) = \int_0^1 t^{a-1}(1-t)^{-a}\mathrm{d}t$$

$t = \sin^2\theta$ here. Then,

$$= 2\int_0^{\pi/2} \sin\theta^{2a-2} \left(1 - \sin^2\theta\right)^{-a} \cdot \sin\theta\cos\theta \mathrm{d}\theta$$

$$= 2\int_0^{\pi/2} \sin^{2a-1}\theta \cos^{1-2a}\theta \mathrm{d}\theta = 2\int_0^{\pi/2} \tan^{2a-1}\theta \mathrm{d}\theta$$

Because $\tan\theta = u$, we have,

$$= 2\int_0^\infty u^{2a-1} \frac{\mathrm{d}u}{\sec^2\theta} = 2\int_0^\infty \frac{u^{2a-1}}{1+u^2}\mathrm{d}u$$

Sub in $u = \sqrt{x}$.

$$= 2\int_0^\infty \frac{x^{2a-1/2}}{1+x}\frac{\mathrm{d}x}{2\sqrt{x}} = \int_0^\infty \frac{x^{a-1}}{1+x}\mathrm{d}x = \frac{\pi}{\sin \pi a}$$

We hence forth showed,

$$\boxed{\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin \pi a}}$$

For $0 < a < 1$ and for $a = \frac{1}{2}$ we have,

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}$$

You can think about

Valid for $\forall a \in \mathbb{C}$ except $a \in \mathbb{Z}$,

$$\Gamma(a)\Gamma(1-a) - \frac{\pi}{\sin a\pi}$$

limit point of 0 function is identically 0. Corollary: $\Gamma(z) \neq 0$.

Formula for Gamma Function

With $\mathrm{Re}(z) > 0$, with,

$$e^{-t} = \lim_{n\to\infty}(1 - \frac{t}{n})^n$$

The proof for this is,

$$-t = \lim_{n\to\infty}\frac{\ln(1-t/n)}{1/n}$$

applying l'Hopital. From that definition of e^{-t} we can hope,

$$\Gamma(z) = \lim_{n\to\infty}\int_0^n t^{z-1}(1-t/n)^n\,\mathrm{d}t$$

Pretty easy to justify because there is an integrable function defined on $(0, \infty)$ which dominates the integral.

Now evaluate,

$$\int_0^n t^{z-1}(1-t/n)^n\mathrm{d}t$$

If we set $t = ns$,

$$\int_0^1 (ns)^{z-1}(1-s)^n n\mathrm{d}s = n^z\int_0^1 s^{z-1}(1-s)^n\mathrm{d}s$$

$$= n^z B(z, n+1) = n^z\frac{\Gamma(z)\Gamma(n+1)}{G(z+n+1)}$$

$$= n^z n!\frac{\Gamma(z)}{(z+n)(z+n-1)\Gamma(z+n-1)}$$

Here $\Gamma(z)$ cancels because of with respect to n times.

$$= \frac{n^z n!}{(z+n)(\cdots)(z+1)z}$$

$$\frac{1}{\Gamma(z)} = \lim_{n\to\infty}\frac{z(z+1)(\cdots)(z+n)}{n^z n!}$$