Computational Complex Analysis : : Class 24

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Continue the discussion on the fact that every non constant holomorphic function is an open function. f be holomorphic function then $f(z_0) = w_0$ N times. Except at z_0, f is not going to be w_0. If we use residue theorem and call the circle around z_0 as C_0 then

$$N = \frac{1}{2\pi i} \int_{C_0} \frac{f'(z)}{f(z) - w_0} \mathrm{d}z$$

N is the order of 0 here in $f(z) = w_0$, where

$$f(z) = c(z - z_0)^N + \dots$$

$c \neq 0$. If $|w - w_0|$ is sufficiently small then

$$|f(z) - w| \geq |f(z) - w_0| - |w - w_0| > 0$$

That integral can be replaced with

$$1/2\pi i \left(\int_{C_0} \frac{f'(z)}{f(z) - w} \mathrm{d}z \right)$$

The denom. is a continuous function of w. N is the number of times here. $f(z)$ has every value close enough to w_0. f is an open function.

Trivial example $f(z) = z^N$. $z_0 = 0$. Look at $f(z) - \epsilon$.

Now a question, why is it that the close enough to w_0 the N values of $f(z) = w$ z close to z_0 are distinct?

Summation of Infinite Series

Summation of infinite series. We are going to look at holomorphic functions f defined on $\mathbb{C}$ and we hope to calculate $\sum_{-\infty}^{\infty} f(n)$. Starting point

$$\operatorname{Res}(\cot \pi z, n) = \frac{\cos \pi n}{\pi \cos \pi n} = \frac{1}{\pi}$$

Multiply both sides by

$$\operatorname{Res}\left(\pi \cot \pi z, n\right) = 1$$

f being holomorphic at n,

$$\operatorname{Res}\left(f(z) \pi \cot \pi z, n\right) = f(n) \operatorname{Res}\left(\pi \cot n\pi\right) = f(n)$$

Take the path. On real number line take $N + \frac{1}{2}$ and $-N - \frac{1}{2}$. Then vertically go way up and at the center is origin on the real line. One side would be at Ni and another at $-Ni$. We will take integral over this square with the origin at the center with side length N.

$$\frac{1}{2\pi i} \int_{C_N} f_(z) \pi \cot \pi z \mathrm{d}z = \sum_{n=1}^{\infty} \text{Residues inside.}$$

So if $N \to \infty$ then $f \to 0$ as assumed, at least quadratically,

$$|f(z)| \leq \frac{C}{N^2} \text{ on } C_N$$

So we hope we will be able to utilize the product $f(z)\pi\cot\pi z$ and we are ought to show that,

$$|\pi\cos\pi z| \leq \text{constant on } C_n$$

Setting $N \to \infty$ and the integral overall turns into,

$$0 = \sum_{-\infty}^{\infty} f(n) + \mathrm{Res}\left(f(z)\pi\cot\pi z, 0\right)$$

So we get,

$$\sum_{-\infty}^{\infty} -\mathrm{Res}\left(f(\pi)\cot\pi z, 0\right)$$

Small example can be

$$f(z) = \frac{1}{z^2}$$

$$\sum \frac{1}{n^2} = -\mathrm{Res}\left(\frac{\pi\cot\pi z}{z^2}\right) = \frac{\pi^2}{3}$$

Summing for $n = 0$ to ∞ would be $\frac{\pi^2}{6}$.