Computational Complex Analysis : : Class 23

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Slight improvement to Rouche's Theorem

Two holomorphic functions f, g around a curve over the domain D. $D \cup C$.

$$|g(z)| \leq |f(z)| \text{ on } C$$

and $f+g$ is never 0 on C. Then, $f+g$ and f have same number of 0 in D. Proof repeated, want to show,

$$\int_C \frac{(f+g)'}{f+g} - f'/f = 0$$

This is re-written as

$$\int_C \frac{g'f - f'g}{(f+g)f} = \int \frac{(\frac{g}{f})'}{1+\frac{g}{f}}$$

$h = g/f$, so

$$\int_C \frac{h'}{1+h}$$

$$= \int_C \frac{d}{dz}\left(\log(1+h)\right) \mathrm{d}z = 0$$

Before the values of $1+h$ couldn't hit the boundary but now they can hit the boundary. But they cannot go around the origin to change the determination of the argument.

Examples

$3e^z - z$ on $|z| \leq 1$ how many 0 are there? On the boundary C, the $|z| = 1$, and $|3e^z| = 3e^x \pm 3e^{-1} = \frac{3}{e} > 1$ there are no zeroes because e^z has no zeros.

Another example.

$$z^4 - 5z + 1$$

How many zeroes are there in the annulus $1 \leq |z| \leq 2$. Let's see at $|z| = 2$. So $|z|^4 = 16$ and $|-5z+1| \leq 11$, and there looks like 4 zeros. In the smaller $|z| = 1$, $5z$ dominates, and there's 1 zero.

Different proof.

$$\frac{1}{2\pi i}\int_C \frac{f' + tg'}{f+tg}\mathrm{d}z = N(t)$$

Definition 1. Suppose f is a function defined on some set in $\mathbb{R}^2$ with values in some set in $\mathbb{R}^2$. We say that f is an *open function* if for every open set G where f is defined f the image of $f(G)$ is also open. $f(x) = e^x$. An open interval in input maps to an open interval in output.

$f(x) = x^2$ is not open.

Theorem 1. Every non constant holomorphic function is an open function.

Proof. Let z_0 be fixed. Then let $w_0 = f(z_0)$. $f(z_0) - w_0$ is 0 at z_0. Let's call the order of zero as $N \geq 1$. i.e.

$$f(z) - f(z_0) = (z - z_0)^N g(z)$$

Where $g(z_0) \neq 0$. There is no sequence $z_k \to z_0$ for which $f(z_k) = w_0$. Therefore there should be a circle centered on z_0 on inside which $f = w_0$ only at z_0. □

Problem for self, does an open interval can map to something closed?