Computational Complex Analysis : : Class 22

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Counting Theorem

Set we have a nice curve with a boundary region. And we have a holomorophic function f with some isolated singularities, namely Poles. We want to count the number of zeroes and number of poles of f. We assume we can use the residue theorem and on the curve we don't have f being infinite or zero.

Now we apply the residue theorem not to f itself but to $f'(z)/f(z)$. Examine

$$\int_C \frac{f'(z)}{f(z)} \mathrm{d}z$$

What are the singularities inside? Wherever f is zero we have singularity. Wherever f is infinite we also have it same. Examine singularities of f'/f.

Let's say $f(z_0) = 0$. Now near z_0,

$$f(z) = (z - z_0)^m g(z)$$

so that $g(z_0) \neq 0$.

$$(z - z_0)^m \left(a_0 + a_1(z - z_0)^1 + a_2(z - z_0)^2 + \cdots\right)$$

Let's compute our quotient

$$\begin{aligned} f'/f &= \frac{(z - z_0)^m g'(z) + m(z - z_0)^{m-1} g(z)}{(z - z_0)^m g(z)} \\ &= \frac{g'(z)}{g(z)} + \frac{mg(z)}{g(z)} \frac{1}{z - z_0} + \cdots \end{aligned}$$

So the residue of $\frac{f'}{f}$ at z_0 is equal to

$$\operatorname{Res}\left(\frac{f'}{f}, z_0\right) = m$$

Now let's apply the same thing for a pole.

$$\begin{aligned} f(z) &= \frac{C}{(z - z_0)^m} + \cdots \\ &= \frac{1}{(z - z_0)^m} g(z) \\ g(z_0) &= C \\ \log f &= -n \log(z - z_0) g + (z - z_0)^{-n} \log g \\ \frac{f'}{f} &= \frac{-ng}{z - z_0} + \cdots \end{aligned}$$

Residue Theorems

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} \mathrm{d}z = \text{ total number of zeroes} - \text{ total number of poles}$$

Let's consider a polynomial $P(z)$ with order $n \geq 1$ and consider the curve to be a circle. If it has zeroes it has finitely many so we just need to draw a big radius.

$$\int_C \frac{1}{2\pi i} \frac{P'(z)}{P(z)} \mathrm{d}z = \mathcal{Z}$$

Rough estimates P_z will have leading term $cz^n +$ lower order if $C \neq 0$. Then $P'(z) = ncz^{n-1}$.

$$P'/P = \frac{ncz^{n-1}(1 + \ldots)}{cz^n(1 + \ldots)} = \frac{n}{z} + \cdots$$

$$\int P'/P \mathrm{d}z = \int \frac{n}{z} \mathrm{d}z = n\frac{2\pi i}{R^{n-1}}$$

$$P(z) \text{ has exactly } n \text{ zeroes in } \mathbb{C}$$

Rouche's Theorem

Back to the general situation. f is defined to be a holomorphic as before and g is also defined holomorphic as before. But assume $|g(z)| < |f(z)|$ on $\mathbb{C}$. $f \neq 0$ on $\mathbb{C}$ and $f + g \neq 0$ on $\mathbb{C}$.

The two functions $f(z)$ and $f(z) + g(z)$ have the same value of number of zeroes minus the number of poles inside $\mathbb{C}$.

To prove the theorem we need to show

$$\int f'/f \mathrm{d}z = \int \frac{f' + g'}{f + g}$$

$$\int_C \left(f'/f - \frac{f' + g'}{f + g} \right)$$

Take common denominator

$$= \frac{f'(f + g) - f(f' + g')}{f(f + g)} = \frac{f'g - fg'}{f(f + g)} = \frac{-\left(\frac{g}{f}\right)'}{f(f + g)}$$

$h = g/f$ gives $= \frac{-h'}{1+h}$. So we now have to see h values on $\mathbb{C}$. $|h| < 1$ from the starting point. We need to prove $\int -\frac{h'}{1+h} \mathrm{d}z$.

$\log h$ is well defined because it's argument $-\frac{\pi}{2} + \frac{\pi}{2}$ and now $\log 1 + h$ has derivative $h'/1 + h$.

$$\int_C \frac{d}{dz} \log(1 + h) = 0$$

This can be used to solve the Fundamental Theorem of Algebra here again.

$$P(z) = cz^n + \cdots$$

$$f(z) = cz^n$$

And this dominates the lower order term on the big circle.