Computational Complex Analysis : : Class 21

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We are going to look at z^α. Both quantity are $\mathbb{C}$.

$$z^\alpha = \exp(\alpha \log z)$$

Note

$$= \exp\left(\alpha\left(\ln|z| + i \arg z\right)\right) = \exp\left(\alpha \ln|z|\right) \exp\left(i\alpha \arg z\right)$$

$z \neq 0$, it's ambiguous. Since $\log z$ is ambiguous. So also we could say z^α is exponential of $\alpha \log z$ but we can add

$$z^\alpha = \exp\left(\alpha\left(\log z + 2\pi i n\right)\right)$$

$$z^\alpha = \exp\left(\alpha \log z + 2\pi i \alpha n\right)$$

If α is an integer we have it defined. z^α equals as it should.

Let's work on how about $\alpha = 0$. How about 1^α,

$$1^\alpha = \exp\left(\alpha i \arg 1\right) = \exp(\alpha i \left(2\pi n\right))$$

It has infinitely many values. α is complex.

Derivative

What about

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(z^\alpha\right) = \frac{\mathrm{d}}{\mathrm{d}z} \exp\left(\alpha \log z\right) = \exp\left(\alpha \log z\right) \frac{\alpha}{z}$$

$$= \frac{\alpha}{z} z^\alpha = \alpha z^{\alpha - 1}$$

So it holds. For another one,

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} = z^\alpha \left(\log z\right)$$

For fun try i^i.

Integral

New computation of an integral. We want to go from 0 to ∞.

$$\int_0^\infty \frac{x^{\alpha - 1}}{x + 1}$$

Set $\alpha \in \mathbb{R}$. The condition on α to make the integral finite as $x \to 0$ and $x \to \infty$.

$$1 > \alpha > 0$$

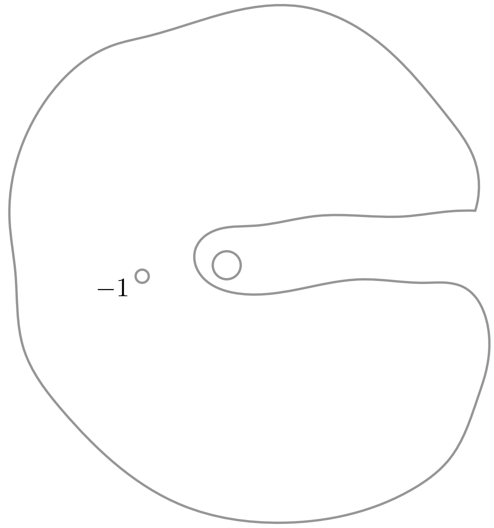

Figure 1: Integral around for Class 21

Just confirms the integral will exist. Now I am going to look at the plane. We are interested in the origin, of course, but we want to go to $+\mathbb{R}$. z has singularity at $z = -1$. We want to use $f(z) = \frac{z^{\alpha-1}}{z+1}$

$$\int_C \frac{z^{\alpha-1}}{z+1} \mathrm{d}z = 2\pi i \,(\text{Residue at -1}) = 2\pi i \frac{(-1)^{\alpha-1}}{1}$$

Then integrand numerator equals

$$\exp\left((\alpha-1)\log z\right) = \exp\left((\alpha-1)\, i \arg(-1)\right) = \exp\left((\alpha-1)\right) i\pi$$

The details are

$$\int_0^\infty \frac{x^{\alpha-1}}{x+1}\mathrm{d}x + \int_\infty^0 \frac{x^{\alpha-1}}{x+1} e^{2\pi i(\alpha-1)}$$

Beware about going around the circle changes the argument by 2π

$$\int \frac{x^{\alpha-1}}{x+1}\mathrm{d}x = \frac{2\pi i e^{\pi i(\alpha-1)}}{1-e^{2\pi i(\alpha-1)}}$$

$$= \pi/\sin\pi\alpha$$

Here the ϵ is a radius that will go to zero around 0.

Handy methods to extend this example

Here's one. Take the formula and change dummy variable,

$$\boxed{\int_0^\infty \frac{x^{\alpha-1}}{x+1}\mathrm{d}x = \pi/\sin\pi\alpha}$$

Using $x = y^3$,

$$\pi/\sin\alpha\pi = \int_0^\infty \frac{y^{3\alpha-3}}{y^3+1} 3y^2 \mathrm{d}y = \int_0^\infty \frac{y^{3\alpha-1}\mathrm{d}y}{y^3+1}$$

$$\frac{\pi}{\sqrt{3}/2} = 3\int_0^\infty \mathrm{d}y \frac{1}{y^3+1} = \frac{2\pi}{3\sqrt{3}}$$

Differentiate the equation

$f;dsakjl;lkjfdsa;lkjfdsa;lkjfdsa;lkjfdsa;lkjfdsaf;lkjdsa;lkjfdsafdsa;lkj;lkjfdsalkjfdsalkjfdsafdsa;lkj$

$$\int_0^\infty \frac{x^{\alpha-1}\ln x}{x+1}\mathrm{d}x = \pi\frac{\mathrm{d}}{\mathrm{d}\alpha}\csc\pi\alpha = -\pi^2\csc\pi\alpha\cot\pi\alpha$$

So for $\alpha = \frac{1}{2}$,

$$\int_0^\infty \frac{x^{-\frac{1}{2}}\ln x}{x+1} = 0$$

For $x = e^y$, then x goes from 0 to ∞ so y goes from $-\infty$ to ∞.

$$\int_{-\infty}^\infty e^y \mathrm{d}y \frac{e^{(\alpha-1)y}}{e^y+1} = \frac{\pi}{\sin\pi\alpha}$$

$$\int_{-\infty}^\infty \mathrm{d}y \frac{e^{\alpha y}}{e^y+1} = \frac{\pi}{\sin\pi\alpha}$$