Computational Complex Analysis : : Class 20

February 23, 2024

Ahmed Saad Sabit, Rice University

We are going to exploit

$$z = re^{i\theta}$$

$$\log z = \ln r + i\theta$$

$$\log z = \ln |z| + i \arg z$$

Today we will look at

$$0 < \arg < 2\pi$$

The function

$$f(z) \log z$$

Residue theorem

$$\frac{1}{2\pi i} \int_C f(z) \log z \mathrm{d}z = \text{ sum of the residues of integrand } f(z) \log z \text{ inside the path}$$

$$\frac{1}{2\pi i} \int_\epsilon^R f(x) \ln x \mathrm{d}x + \frac{1}{2\pi i} \int_{-C} \cdots + \frac{1}{2\pi i} \int_R^\epsilon f(x)(\ln x + 2\pi i) \mathrm{d}x$$

Taking the limit

$$0 = \frac{1}{2\pi i} \int_0^\infty f(x) \ln x - \frac{1}{2\pi i} \int_0^\infty f(x) \left(\ln x + 2\pi i \right) \mathrm{d}x = \text{ sum of residues}$$

$$\boxed{\int_0^\infty f(x) \mathrm{d}x = - \sum f(z) \log z \text{ residues}}$$

Examples

$$\int_0^\infty \frac{\mathrm{d}x}{x^3 + 1}$$

so this is going to be

$$- \sum_n \text{residues of } \frac{\log z}{z^3 + 1}$$

Singularities happen at $z^3 = -1$

$$z = e^{\frac{\pi i}{3}}, e^{\pi i}, e^{\frac{5\pi i}{3}}$$

$$\text{res} \left(\frac{\log z}{z^3 + 1} \right) = \frac{\log z}{3z^2} = \frac{z \log z}{3z^3} = - \frac{z \log z}{3}$$

Look at

$$z^\alpha$$