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The integral we are going to do today is similar but mostly different. In every complex analysis book you will find

$$\int_{-\infty}^{\infty} \frac{\sin ax}{x} \mathrm{d}x$$

Using a dummy variable we can do a rewrite,

$$\int_{-\infty}^{\infty} \frac{\sin t}{t/a} \mathrm{d}t/a$$

So in our analysis we might just do the case that $a = 1$ because for other a we might have minor trouble with $-/+$ signs. So we are deal with now is

$$\boxed{\int_{-\infty}^{\infty} \frac{\sin x}{x} \mathrm{d}x}$$

Choose $f(z) = \frac{\sin z}{z}$. Instead, we deal with

$$f(z) = \frac{e^{iz}}{z}$$

This is undefined at 0.

$$|e^{iz}| = e^{\mathrm{Re}(iz)} = e^{-y}$$

Now having a singularity at $z = 0$ causes us to consider a sub circle path of ϵ radius inward to the circle and we get

$$\int_C \frac{e^{iz}}{z} \mathrm{d}z = 0$$

We need $R \to \infty$ and $\epsilon \to 0$. Let's talk about $R \to \infty$ first.

So we are going to integrate along $\bigcap$ path and taking modulus

$$\left| \int e^{iz}/z \mathrm{d}z \right| \leq \int \frac{|e^{iz}||\mathrm{d}z|}{|z|}$$

From here we get,

$$= \int_C \frac{e^{-y} R \mathrm{d}\theta}{R} = \int_C e^{-y} \mathrm{d}\theta$$

Using $y = R \sin \theta$

$$= \int_0^{\pi} e^{-R \sin \theta} \mathrm{d}\theta$$

The problem is this integral tries to go to infinitely zero, other than the two points $\theta = 0$ and $\theta = \pi$. Using symmetry we can go from

$$2 \int_0^{\pi/2} e^{-R \sin \theta} \mathrm{d}\theta$$

This is smaller than

$$< 2 \int e^{-R\pi\theta/2} \mathrm{d}\theta$$