Computational Complex Analysis : : Class 18

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$$\int_0^\infty \frac{\mathrm{d}x}{x^4 + x^2 + 1}$$

$$\int_{-\infty}^\infty \frac{\cos ax}{x^2 + 1}\,\mathrm{d}x$$

$$\int_0^\infty \frac{\mathrm{d}x}{x^3 - i}$$

Residue theorem says,

$$\frac{1}{2\pi i}\oint f(z)\mathrm{d}z = \sum \text{Residue inside } C$$

Illustration of a Property

One of the properties of residues,

$$\mathrm{Res}\left(fg', z_0\right) = -\mathrm{Res}(f'g, z_0)$$

I want to calculate the residue of

$$\mathrm{Res}\left(\csc^3(z), 0\right)$$

We know that

$$\mathrm{Res}\left(\csc(z), 0\right) = 1$$

You get a homework assignment to find,

$$\mathrm{Res}\left(\csc^n(z), 0\right)$$

Here n is odd otherwise we have an even function.

We will show how to go from first power to the third power. Similar technique for the homework problem.

$$\mathrm{Res}\left(\csc^3\right) = \mathrm{Res}\left(\csc \cdot \csc^2\right)$$

We can leave out the origin, and csc is a derivative of something.

$$\tan' x = \sec^2 x \quad \text{and} \quad \cot' x = -\csc^2 x$$

We can find, using the given mentioned formula,

$$= \mathrm{Res}\left(\csc \cdot (-\cot')\right) = \mathrm{Res}\left(\csc' \cot\right)$$

$$= \mathrm{Res}\left(-\csc \cot \cot\right)$$

$$= \mathrm{Res}\left(-\csc \cot^2\right)$$

Y'all knew the rule $\sec^2 = 1 + \tan^2$

$$= \mathrm{Res}\left(-\csc^3 + \csc\right) = -\mathrm{Res}\left(\csc^3\right) + \mathrm{Res}\left(\csc\right)$$

Left hand and right hand written together,

$$\text{Res}\left(\csc^3\right) = -\text{Res}\left(\csc^3\right) + \text{Res}\left(\csc\right)$$

We get

$$2\text{Res}\left(\csc^3\right) = \text{Res}(\csc) = 1$$

Hence we get,

$$\boxed{\text{Res}\left(\csc^3\right) = \frac{1}{2}}$$

Calculations

$$\int_0^\infty \frac{\mathrm{d}x}{x^4 + x^2 + 1}$$

$$f(z) = \frac{1}{z^4 + z^2 + 1}$$

Our integral goes from two sides of infinity. That integral is bound from 0 to ∞, so we just need half of the infinite as that's an even function

$$= \frac{1}{2}\int_{-\infty}^\infty \frac{\mathrm{d}x}{x^4 + x^2 + 1}$$

There are some poles in the semi-circle region we considered.

$$\frac{1}{2\pi i}\int_C f(z)\mathrm{d}z = \sum \text{Res}\left(f(z), \text{upper half plane}\right)$$

An extra step, multiply both sides of the function with $z^2 - 1$

$$f(z) = \frac{z^2 - 1}{z^6 - 1}$$

So the roots are

$$z^6 = 1 = e^{2n\pi i}$$
$$z = e^{\frac{2n\pi i}{6}}$$

n here goes from $0, 1, 2, 3, 4, 5$. Testing over `a \over b`

$$a \over b$$

The roots in the upper half plane are

$$e^{2\pi i \frac{1}{6}} = e^{2\pi i/6}, e^{4\pi i/6}$$

"How did we get the $e^{3\pi i \frac{1}{6}}$? Prof: By mistake".

Now the integral over the "path" (not "region") is

$$\frac{1}{2\pi i}\int_C f(z)\mathrm{d}z = \text{Res}\left(f, e^{\pi i \frac{1}{3}}\right) + Res(e^{2\pi i/3})$$

Now we need to check what f is like over R for $R \to \infty$, we need to have f tends to zero hence the limit gives me a basically 0.

$$\text{Res}\left(\frac{z^2 - 1}{z^6 - 1}, z_0\right) = \frac{z_0^2 - 1}{6z_0^5} = \frac{z_0^3 - z_0}{6}$$

The z_0 are $e^{\frac{2\pi i}{6}}, e^{\frac{4\pi i}{6}}$

$$\frac{1}{2}\int_{-\infty}^\infty f(x)\mathrm{d}x = \frac{1}{2}2\pi i\left(\frac{e^{\pi i} - e^{\frac{\pi i}{3}}}{6} + \frac{e^{2\pi i} - e^{\frac{2\pi i}{3}}}{6}\right) = \boxed{\frac{\pi}{2\sqrt{3}}}$$

Another Computation

$$\boxed{\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1} \mathrm{d}x}$$

Here's a function of z and here

$$f(z) = \frac{\cos az}{z^2+1}$$

We cannot use $a = i\alpha$ because that blows up. Hence we change it a bit,

$$f(z) = \frac{e^{iaz}}{z^2+1}$$

We have to pick the real portion for $\cos az$. Here $a \in \mathbb{R}$. Investigate what happens when for the semicircle $R \to \infty$. Now

$$|e^{iaz}| = e^{\mathrm{Re}(iaz)} = e^{\mathrm{Re}(iax-ay)} = e^{-ay}$$

Now we are required to say a is $a > 0$ otherwise we have something blowing up. Hence

$$a \geq 0$$

Back to the function now,

$$|f(z)| = \left| \frac{e^{iaz}}{z^2+1} \right|$$

Now we need, where the singularities happen? Well at $z = i$ for upper half plane.

$$\mathrm{Res}\left(\frac{e^{iaz}}{z^2+1}, i\right) = \frac{e^{-a}}{2i}$$

For $R \to \infty$, the residue theorem

$$\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{e^{iax}}{x^2+1} \mathrm{d}x = \frac{e^{-a}}{2i}$$

$$\boxed{\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1} = \pi e^{-a} \qquad (a \geq 0)}$$

$$R \quad \mathbf{R} \quad \boldsymbol{R}$$

Question: Is it practical to solve for definite integrals given we knew how the function behaved at finite $\boldsymbol{R}$?