Computational Complex Analysis : : Class 17

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Ahmed Saad Sabit, Rice University

Residue Theorem

Apply Cauchy's theorem if f has poles as shown. Let there z_1, z_2, z_3 each are there. We apply the cauchy's theorem on a region that doesn't have the neighborhood around the singularities.

$$\int_{\text{region}} f(z)\mathrm{d}x\mathrm{d}y = \frac{1}{2\pi i}\int_{\text{boundary}} f\mathrm{d}z$$

For the region with holes being analytic

$$\frac{1}{2\pi i}\int_{\text{boundary}} f(z)\mathrm{d}z = 0$$

$$\int_{\text{boundary of } D} f(z)\mathrm{d}z + \sum_{\text{CW circles}} \int f(z)\mathrm{d}z = 0$$

$$\int_{\partial D} f(z)\mathrm{d}z = \sum \int_{CCW} f(z)\mathrm{d}z$$

$$\frac{1}{2\pi i}\int_{\partial D} f(z)\mathrm{d}z = \sum_{z\in D}\mathrm{Res}(f, z)$$

Example

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2+1} =$$

Calc 2, what do we do? Take $\arctan x$ from $-\infty$ to ∞. We get

$$\frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$

Let's do this using the residue theorem. And then we can move on to the residue theorem x^2 to x^4.

So I define $f(z) = \dfrac{1}{z^2+1}$. It has poles at i and $-i$.

$$\mathrm{Res}(f, i) = \frac{1}{2z}(z = i) = \frac{1}{2i}$$

$$\mathrm{Res}(f, -i) = \frac{1}{2z}(z = -i) = -\frac{1}{2i}$$

As

$$\mathrm{Res}\frac{a(z)}{b(z), z_0} = \frac{a(z_0)}{b'(z_0)}$$

Now we have to find a region. The region will be semicircle in complex plane with one of the poles contained. Radius of this semicircle sunset be

$$\frac{1}{2\pi i}\int_{\partial D_R}\frac{1}{z^2+1}\mathrm{d}z = \mathrm{Res}(f,i) = \frac{1}{2i}$$

$$\int_{\partial D_R}\frac{\mathrm{d}z}{z^2+1} = \pi$$

We can estimate the bounds

$$|\int_{\cap}\int f(z)\mathrm{d}z| \leq \int_{\cap}|f(z)\mid \mathrm{d}z| = \int\frac{1}{|z^2+1|}|\mathrm{d}z| \leq \int_0^{\pi}\frac{1}{R^2-1}R\mathrm{d}\theta$$

Example

Getting even more serious

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{x^4+1}$$

Choose the holomorphic function

$$f(z) = \frac{1}{z^4+1}$$

Find residues,

$$z^4 = -1 = e^{\pi i} = e^{3\pi i} = e^{5\pi i} = e^{7\pi i}$$

$$z = e^{\frac{\pi i}{4}}, e^{3\pi\frac{i}{4}}, e^{\frac{5\pi i}{4}}, e^{\frac{7\pi i}{4}}$$

Now let's find the residues for each

$$\mathrm{Res} = \frac{1}{4z^3} = \frac{z}{4z^4} = -\frac{z}{4}$$

As $z^4 = -1$.

Now, choose the path.

Now apply the residue theorem

$$\frac{1}{2\pi i}\int_{-R}^{R} + \int_{\cap} f(z)\mathrm{d}z = \sum \text{residues of 2 points inside the region} = \frac{-e^{\pi i\frac{1}{4}}}{4} - \frac{e^{\frac{3\pi i}{4}}}{4}$$

Computation

$$\frac{\pi}{\sqrt{2}}$$

So the integral is done.

$$f(z) = \frac{1}{z^4+1} = \frac{\pi}{\sqrt{2}}$$

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{x^6+1}$$

The poles occur at

$$z^6 = -1 = e^{\pi i} = e^{3\pi i} = e^{5\pi i} = e^{7\pi i} = e^{9\pi i} = e^{11\pi i}$$

The roots are gotten by simply taking the 6 th root. We choose the path semicircle that is going to have 3 poles in the upper plane. These 3 poles are the upper plane existing roots of the number.

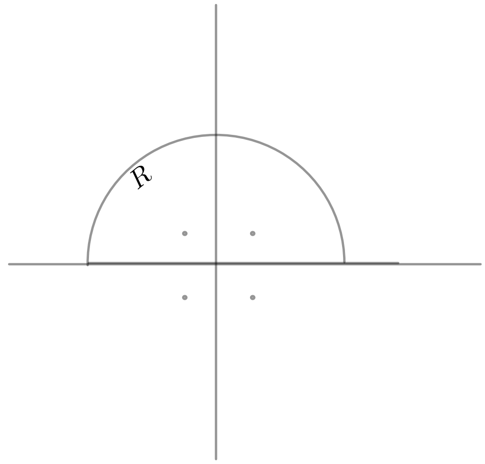

Figure 1: Residue path of a semicircle

$$\int \frac{\mathrm{d}z}{z^6+1} = \sum \text{Residues at 3 points}$$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^6+1}$$

Residues

$$\operatorname{Res}(f,\cdot) = \frac{1}{6z^9} = \frac{z}{6z^6} = -\frac{z}{6}$$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^6+1} = \frac{-e^{\pi \frac{i}{6}} - i - e^{5\pi \frac{i}{6}}}{6} = -\frac{i}{3}$$

So we get

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^6+1} = \frac{2\pi}{3}$$