Computational Complex Analysis : : Class 15

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Given some function which is undefined at a point z_0, we sometimes call z_0 a singularity of the function. We are now going to look at Holomorphic functions f with a singularity at z_0. We are going to define around z_0 but not z_0. And specifically we want to discuss isolated singularities. And that means

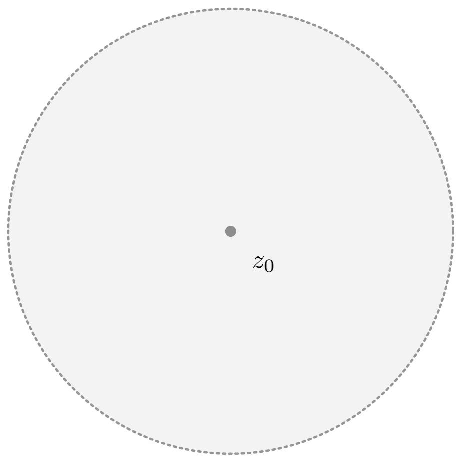

Figure 1: function has to be holomorphic around but not at the point

Say $z = \frac{1}{n\pi}$. Then $\sin \frac{1}{z} = 0$. Now look at $\frac{1}{\sin \frac{1}{z}}$. Every single point other than the origin then becomes an isolated point that is a singularity.

Now we are going to classify the isolated singularities of holomorphic functions. So we will see that there are three classes. Suppose z_0 is an isolated singularity of f. Here's every where the function is holomorphic other than z_0. There is a unique Laurent's Expansion of f in a neighborhood of z_0.

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

This converges for $0 < |z - z_0| < r$.

$$c_n = \frac{1}{2\pi i} \int_C \frac{f(z_0 + s)}{s^{n+1}} \mathrm{d}s$$

Case 01: c_n with $n < 0$ are 0.

$$f(z) = c_0 + c_1(z - z_0) + \dots$$

Riemann's Removable singularity theorem, if the modulus of f is bounded near z_0 then removable.

Case 02: At least one c_n is not 0 with $n < 0$. In fact only finitely many c_n are of this nature.

$$f(z) = c_{-m}/(z - z_0)^m + \cdots + c_0 + c_1(z - z_0) + \cdots$$

What is the limit of $|f(z)|$ as $z \to z_0$? Well it goes to infinity because $\dfrac{C_m}{(z - z_0)^m}$ blows up to infinity. This is called a Pole.

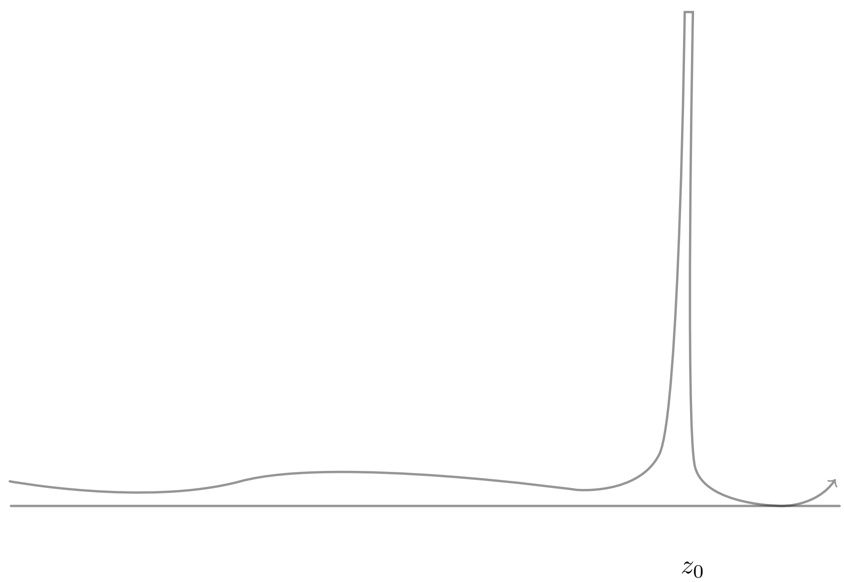

Figure 2: A visual representation of a pole

Case 03: Infinitely many c_n with $n < 0$ are not zero. This is called "Essential Singularity".

Casorati Weierstrass Theorem

Theorem 1. Suppose z_0 is an essential singularity of f. Suppose w is any complex number or ∞. Then the conclusion is there exists a sequence

$$\zeta_1, \zeta_2, \zeta_3, \ldots, \zeta_k$$

Here as $\lim_{k\to\infty} |\zeta_k| = z_0$ and

$$\lim_{k\to\infty} f(\zeta_k) = w$$

If you think of $e^{\frac{1}{z}}$ then

Case 01: $w = \infty$. By contradiction assume there is such sequence. Then $f(z)$ has the property that it cannot have very large as $z \to z_0$.

$$\exists C > 0 \text{ such that } |f(z)| < C \text{ for all } z \text{ close to } z_0$$

From Riemann's Removable Singularity Theorem, z_0 is a removable singularity for f.

Case 02: $w \in \mathbb{C}$. Again, if no such sequence exists, then there is a small disk containing w which is never reached by any value of $f(z)$ near z_0.

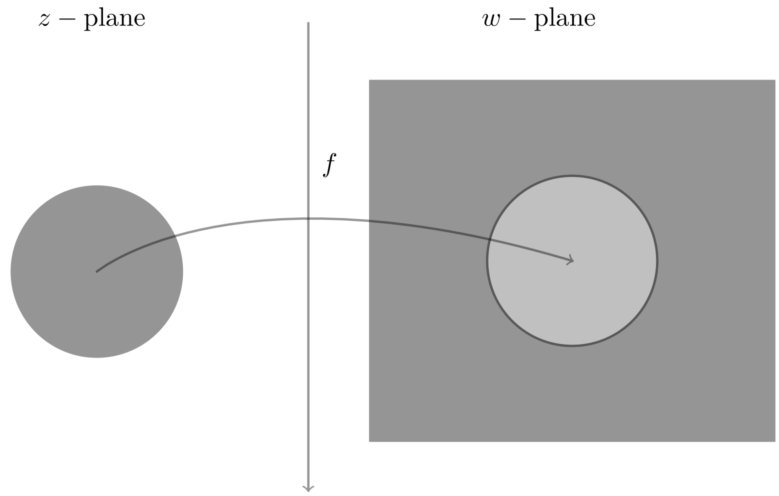

Figure 3: z plane and w plane

For $|f(z) - w| \geq a$ for all $|z - z_0| < r$,

$$\left|\frac{1}{f(z) - w}\right| \frac{1}{a}$$

z_0 is removable singularity $\dfrac{1}{f(z) - w'}$.

This function extends

$$\frac{1}{f(z) - w}$$

is $g(z)$. g is holomorphic so it maybe 0 at z_0. Factor out $(z - z_0)^m$ from g :

$$g(z) = (z - z_0)^m h(z)$$

And

$$h(z_0) \neq 0$$

$$\frac{1}{f(z) - w} = (z - z_0)^m h(z)$$

$$f(z) - w = (z - z_0)^{-m} \frac{1}{h(z)}$$

Laurent Expansion. It's a pole. Contradiction the essential singularity of f at z_0. We did all the work to prove it wasn't an essential singularity instead it was a pole.

We are about to start a huge transition: Look at holomorphic function with isolated singularity at z_0 and look at the Laurent Expansion of

$$f(z) = \cdots c_{-2}(z - z_0)^{-2} + c_{-1}(z - z_0)^{-1} + c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \cdots$$

c_{-1} is important and it's called the residue of f at z_0.

$$\boxed{c_{-1}}$$

What we are going to use this for is called Residue Theory.