Computational Complex Analysis : : Class 13

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We showed in the last class that every holomomorphic functions defined on an open set is also analytic.

The theorem about the zeroes of an analytic function. $\sin \frac{1}{z}$ is holomorphic, analytic on the complex plane. But without the origin. And where are it's zeros? $\frac{1}{z} = n\pi$. A power series centered at z_0 cannot be 0 on a sequence converging to z_0.

Theorem 1. If f is holomorphic on a connected open set, the zeroes of f cannot have a limit point in the open set.

Proof. By contradiction we consider a sequence

$$z_1, z_2, \dots, f(z_L) = 0$$

such that the sequence converge and

$$\lim_{n\to\infty} z_n \in \text{open set}$$

One limit point of zeroes then epidemically spreads to make the whole sequence become zero. □

Commented „Very powerful". New proof to $e^{x+y} = e^x e^y$.

- Calc 2 gives $e^{x+y} = e^x e^y$.
- Let $x \in \mathbb{R}$ be fixed and let $f(x) = e^{x+z} - e^x e^z$. For f fixed for $\mathbb{R}$ we have 0. We set a theorem $f = 0$ for $z \in \mathbb{C}$.
- Let $z \in \mathbb{C}$ be fixed and consider the holomorphic function of w.

  $$e^{w+z} = e^w e^z$$

  The function $e^{x+z} - e^x e^z = 0$ for $x \in \mathbb{R}$.

Best version of maximum modulus principle.

Theorem 2. Suppose f is holomorphic on a connected open set and at some z_0,

$$|f(z)| \leq |f(z_0)|$$

For all z in some neighborhood of z_0. Conclusion f is constant. Maximum modulus principle says the disk says is constant and we can epidemically make it spread. Theorem, suppose f is holomorphic in a connected open set and $|f|$ has local minimum at some point in the set then $f = 0$.

Theorem 3. Suppose f is holomorphic on all of $\mathbb{C}$. And suppose that the limit

$$\lim_{z\to\infty} |f(z)| \text{ is } \infty$$

We just say the function grows with radius. Now set $f(z) = 0$ for some z. I am thinking the origin point doesn't matter.

Proof. Proof by contradiction, suppose, the contradiction would be $f(z)$ is never 0. Then $\frac{1}{f}$ is holomorphic on the whole plane. And $\frac{1}{f}$ and since $|\frac{1}{f}| \to 0$ as $z \to \infty$, $1/f$ has a minimum. So $\frac{1}{f} = 0$ at some point and this is a contradiction. □

Fundamental Theorem of Algebra

Every polynomial of $f(z)$ of positive degree is zero at some point. Proof $f(z) = Cz^n + \ldots$.

Another proof of Liouvilles Theorem,

$$f \text{ holomorphic on } \mathbb{C}$$

and $|f(z)| \leq M \quad \forall z$ and f is constant. Assume f is never 0, then $\frac{1}{f}$ is holomorphic and tends to 0 at infinity.