Computational Complex Analysis : : Class 12

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Maximum Modulus Principle

If f is holomorphic on a connected open set, and if x_0 is in the set, and

$$\mid f(x) \mid \leq \mid f(x_0) \mid$$

for all x in the set, then f is constant.

Proof was shown in Friday. It's the same one that spread like an epidemic.

Corollary

Suppose f is continuous on a closed bounded set of the complex plane and holomorphic on the interior of that set. Then we know that the modulus of $f(z)$ attains a maximum value $|f(z)|$. Any real value continuous function on a closed bonded set obtains it's minimum value and maximum value (it's a fact of calculus 01 if in one dimension).

Minimum Modulus Principle

Suppose f is holomorphic on a connected open open set and $|f|$ attains its minimum value at z_0 :

$$|f(z)| \geq |f(z_0)| \quad \forall z$$

If $|f(z_0)| > 0$ then $\frac{1}{f}$ satisfies the maximal condition. Hence f is constant.

Remark: $|f(z)|$ can attain a minimum value in the set and not be constant $|f(z_0)| = 0$.

Corollary

Similar.

A cool theorem

Theorem 1. Every holomorphic function is analytic.

Proof. Assume f is holomorphic on an open set, and let z_0 be a point in that set. Consider a region and a point z_0, and draw circle centered at z_0 of positive radius that is still contained in the open set. Any small one that doesn't hit the boundary. Call the circle as γ centered at z_0. Fact known is the taylor series looks like for

$$g(z) = \sum_{n=1}^{\infty} c_n (z - z_0)^n$$

Where $c_n = \frac{g^{(n)}(z_0)}{n!}$. We will use the Cauchy Integral Theorem

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(s)}{s - z} \mathrm{d}s$$

Call z such that $|z - z_0| < r$. We have seen

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(s)}{(s - z)^{n+1}} \mathrm{d}s$$

$$\frac{1}{s - z} = \frac{1}{(s - z_0) - (z - z_0)}$$

$$= \frac{1}{s - z_0} \frac{1}{1 - \frac{z - z_0}{s - z_0}}$$

Geometric Series

$$\frac{1}{s - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{s - z_0} \right)^n$$

$$\frac{1}{s - z_0} \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(s - z_0)^{n+1}}$$

Therefore the first thing to do is to move the summation symbol outside.

$$\frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{\gamma} \frac{f(s)(z - z_0)^n}{(s - z_0)^{n+1}} \mathrm{d}s$$

$$= \frac{1}{\pi 2i} \sum_{n=0}^{\infty} (z - z_0)^n \int_{\gamma} \frac{f(s)}{(s - z_0)^{n+1}} \mathrm{d}s$$

□