Computational Complex Analysis : : Class 11

February 2, 2024

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Some Review

Definition 1. A complex valued function defined from open subset of a complex plane is holomorphic if it has a complex derivative at every point. And f' is continuous. In Brown and Churchill they don't mention but it's automatically proven there in the book using Goursat's Theorem.

Definition 2. A complex valued function defined on an open set is analytic if at each point, say z_0, it agrees with the Taylor's series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

In the neighborhood of z_0. This is automatically holomophic because you can take integral how many times you want.

Theorems mentioned in Class

1

At any point of a function's domain if we can take derivative then we will call it holomorphic. If we can turn it into a series then it is analytic.

If f is real valued and holomorphic (that means you can take derivative anywhere) then f is constant. Take the Cauchy Riemann equation and see that one side is purely real and other side is purely imaginary then the function is just a constant real valued function. (f is real valued already, then the imbalance happens.)

2

If f is holomorphic and $|f|$ is constant, then f is constant.

$$|f|^2 = f\overline{f}$$

Taking $\frac{\partial}{\partial x}$ we get

$$0 = f_x\overline{f} + f\overline{f_x}$$

Similarly for $\frac{\partial}{\partial y}$,

$$0 = f_y\overline{f} + f\overline{f_y}$$

This can be rewritten as

$$0 = \frac{f_y}{i}\overline{f} - f\overline{\frac{f_y}{i}}$$

Hence

$$f_x = \frac{f_y}{i} = -\frac{f_y}{i}$$
$$f_y = f_x = 0$$

3

Cauchy Integral formula

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{s - z} \, \mathrm{d}s$$

4

Maximum modulus principle

Theorem 1. This is quite important for holomorphic functions. It says suppose f is holomorphic on a connected open set. And suppose z_0 the value of the modulus of $|f(z_0)|$ is $\geq$ of all $|f(z)|$ in the neighborhood of z_0.

Connected means I can go to one set to the other with a path contained in the set.

Then f is constant! You cannot have mountain peaks in Holomorphic functions.

Proof. Let's use the Cauchy Integral formula on a disk centered at z_0. Consider this disk to be small. Imagine a point z_0 and there is a ball around it of radius ϵ (I don't care). Now I am going to use the cauchy

$$f(z_0) = \frac{1}{2\pi i} \int_{|s-z_0|=r} \frac{f(s)}{s - z_0} \, \mathrm{d}s$$

We will find the $f(z_0)$ only using the circle around z_0. Consider $s = z_0 + re^{i\theta}$ where $\theta \in [0, 2\pi]$.

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} rie^{i\theta} \mathrm{d}\theta$$

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) \mathrm{d}\theta$$

Now considering the absolute value,

$$|f(z_0)| = |\frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) \mathrm{d}\theta| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{i\theta})| \mathrm{d}\theta$$

$$0 \leq \frac{1}{2\pi} \int_0^{2\pi} (|f(z_0 + re^{i\theta})| - |f(z_0)|) \mathrm{d}\theta$$

$|f(z_0)|$ is greater than or equal to it's neighborhood so up there we end up getting negative values which is a contradiction. So it can only be equal to 0. This proves f has constant value.
This only proves for this neighboring region to z_0. But let's consider another point inside the limit that is near by z_0 but not z_0 itself. Consider neighborhood of that function. Like an epidemic it will spread.
So f is constant everywhere. □

Proof of $\int_a^b g(t) \mathrm{d}t = I$ for g complex valued, and to show that the modulus of I is less than or equal to the modulus of the integral of $|g(t)|$.

Proof.

$$I = |I|e^{i\theta}$$

We will write

$$|I| = e^{-i\theta}\int_a^b g(t)\mathrm{d}t$$

We didn't do anything random. Pushing that into the integral sign,

$$= \int_a^b e^{-i\theta}g(t)\mathrm{d}t$$

You can do this for θ being constant. The left side is purely real. But the right side function is not necessarily real. Now only integrating the real part is enough,

$$= \int_a^b \mathrm{Re}(e^{-i\theta}g(t))\mathrm{d}t$$

We can replace this with a larger real valued function

$$= \int_a^b \mathrm{Re}(e^{-i\theta}g(t))\mathrm{d}t \leq \int_a^b |e^{-i\theta}g(t)|\mathrm{d}t = \int_a^b |e^{-i\theta}||g(t)|\mathrm{d}t$$

$$= \int_a^b |g(t)|\mathrm{d}t$$

□

1 Liouville's Theorem

Suppoe f is holomorphic on all of $\mathbb{C}$ and suppose f is bounded.

$$|f(z)| \leq M$$

For all z. Then the amazing conclusion is that f itself is constant.

Proof. I (Frank Jones) want to show that $f'(z) = 0$ with $\forall z$. Consider a z and saying $f'(z)$ is zero there. Cauchy Integral Formula

$$f(z) = \frac{1}{2\pi i}\int_{|s-z|=R} \frac{f(s)}{s-z}\mathrm{d}s$$

$$f'(z) = \frac{1}{2\pi i}\int_{|s-z|=R} \frac{f(s)}{(s-z)^2}\mathrm{d}s$$

$$|f'(z)| \leq \frac{1}{2\pi}\int \frac{|f(s)|}{|s-z|^2}|\mathrm{d}s|$$

Limmerick

$$s = z + re^{i\theta}$$

$$\mathrm{d}s = ire^{i\theta}\mathrm{d}\theta$$

$$|\mathrm{d}s| = r\mathrm{d}\theta$$

So we have

$$\leq \frac{1}{2\pi}\int_0^{2\pi} \frac{M}{R^2}R\mathrm{d}\theta$$

$$= \frac{M}{R}$$

R can be as large as I please and $f'(z) = 0$

□