z plane can have a point z and look at a direction around the point in z. So, $z + th$ can be a direction around z. Here $h \in \mathbb{C}, t \in \mathbb{R}$. We will get a linear map to w where $w = f(z)$ and $f'(z) = Re^{i\theta}$.

$$f(z+h) = f(z) + Re^{i\theta}h + \dots$$

We are going to look at $f(z+th)$ so that t is small and

$$f(z) + tRe^{i\theta}h$$

One arrow in z is also another arrow in w. Here the direction is going to be $Re^{i\theta}h$. The angles are going to be the same.

1 Review

Let's have a region D that might have holes in it. Then the line integral around γ is

$$\int_\gamma f(z)\mathrm{d}z = 0$$

For all loops γ if and only if f is differentiable. And is continuous, that is Green's theorems. The Cauchy Integral theorem.

2 Cauchy Integral Formula

Assume the same thing that f is differentiable at every point of D and f' is continuous. Here's a daring move - let z_0 be a point $z_0 \in D$ that is not going to change. And try to apply a Cauchy integral theorem to $\frac{f(z)}{z-z_0}$, everywhere other than z_0. And see what Cauchy could do with it. What might I do to modify the situation?

We can't have z_0 in the area of D because the integral is going to be undefined while we are taking the integral at z_0. So z_0 being at the interior, we imagine a disk of radius ϵ around z_0 that is going to be deleted.

We obtain

$$\int_{\partial D_\epsilon} \frac{f(z)}{z-z_0}\mathrm{d}z = 0$$

And $D_\epsilon = D \setminus$ safety disk.

$$\int_{\partial D} \frac{f(z)}{z-z_0}\mathrm{d}z + \int_{\partial \text{ safety disk (clock-wise)}} \frac{f(z)}{z-z_0}\mathrm{d}z = 0$$

$$\int_{\partial D} \frac{f(z)}{z-z_0} = \int_{\partial \text{safety disk, CCW}} \frac{f(z)}{z-z_0}\mathrm{d}z$$

Parametrize $z = z_0 + \epsilon e^{i\theta}$. Here $0 \leq \theta \leq 2\pi$.

$$\int_{\partial D} \frac{f(z)}{z-z_0}\mathrm{d}z = i\int_0^{2\pi} f(z_0 + \epsilon e^{i\theta})\mathrm{d}\theta$$

Left hand side does not depend on ϵ. In spite of what it looks like.

$$\lim_{\epsilon \to 0} i\int_0^{2\ \pi} f(z_0)\mathrm{d}\theta = f(z_0)2\pi i$$

$$\boxed{f(z_0) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(z)}{z-z_0}\mathrm{d}z}$$

Cauchy integral formula, same assumptions on D and f having f being continuous on $D \cup \partial D$ $f'(z)$ exists for all $z \in D$.

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - z_0} \mathrm{d}z$$

Here z becomes the dummy variable and use s instead.

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(s)}{s - z} \mathrm{d}s$$

Observation is we have differentiate both sides with respect to z

$$f'(z) = \frac{1}{2\pi i} \int_{\partial D} f(s) \frac{\mathrm{d}}{\mathrm{d}z} \left(\frac{1}{s - z} \right) \mathrm{d}s$$
$$= \frac{1}{2\pi i} \int_{\partial D} f(s) \frac{1}{(s - z)^2} \mathrm{d}s$$
$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial D} f(s) \frac{\mathrm{d}s}{(s - z)^{n+1}}$$

The math looks cute kintu kisu toh bujtesina :)

Line integral is path independent if it has an anti-derivative. Wait what.

Morera's Theorem follows

Theorem 1. For f continuous on D and suppose and line integral of f along loops in D are always 0 implies that f' exists at every point.

Proof. We already know that $\forall$ differentiable F there is $f = F'$. But now we know that F'' exists, hence f' exists. □

Definition 1. In older days this was called Analytic.
Important Terminology, a function defined on an open set $D \subset \mathbb{C}$ which is differentiable at every point of D is called Holomorphic.

Definition 2. A function f is said to be analytic if near endpoint z_0 of it's domain, it equals a series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

With positive radius of convergence.