Computational Complex Analysis : : Class 09

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Open connected set $D \in \mathbb{C}$ continuous f defined on D. Then these two statement are equivalent:

- The line integral $\int_\gamma f \, \mathrm{d}z$ are equivalent only on $z_0 \neq z_1$ independent of path.
- f has an anti derivative F defined D. F is differentiable on D and $F' = f$. Proof that existence of an anti derivative is given.

$$\int_\gamma f \mathrm{d}z; \int_\gamma F' \mathrm{d}z$$

Here $\mathrm{d}z = \gamma'(t)\mathrm{d}t$

$$= \int_a^b F'(\gamma(t))\gamma'(t)\mathrm{d}t$$

$$= \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\left(F(\gamma(t))\right)\mathrm{d}t$$

ETC

$$F(\gamma(t))_{t=a}^{t=b} = F(z_1) - F(z_0)$$

Converse, we assume the integral of f and

$$\int_\gamma f(z)\mathrm{d}z$$

is an independent path. We want to produce an anti-derivative F. Clever way define $F(z) = \int_w^z f\mathrm{d}z$, this makes sense because any path can do. Try to analyze $F(z+h) - F(z)$. Which is a straight line.

$$F(z+h) = \int_{\text{that path}} f\mathrm{d}z = \int_w^z f\mathrm{d}z + \int_z^{z+h} f\mathrm{d}z = F(z) + \int_z^{z+h} f\mathrm{d}z$$

Parametrize $z + th$

$$F(z+h) - F(z) = \int_0^1 f(z+th)h\mathrm{d}t$$

I am kind of out of clue what is Prof Frank doing at this point. If anything, I want some rest and my head is really heavy right now.

$$\frac{F(h+z) + F(z)}{h} = \int_0^1 f(z+th)\mathrm{d}t$$

The average of $f(z+th)$ at $0 \leq t \leq 1$.

$$F'(z) = f(z)$$

1 Green's Theorem

Let there be a region, and we have a function $f(x,y)$ defined for that region. The region is D. And $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are continuous. Greens theorem says,

$$\iint \frac{\partial f}{\partial x}\mathrm{d}x\mathrm{d}y = \int_{\partial D} f\mathrm{d}y$$

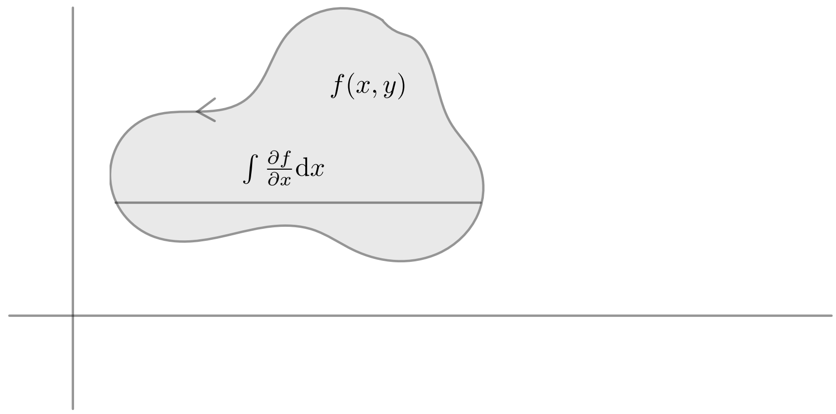

Figure 1: Greens theorem

∂D is the boundary of D. Likewise

$$\iint_D \frac{\partial g}{\partial y}\mathrm{d}x\mathrm{d}y = -\int_{\partial D} -g\mathrm{d}x$$

Now for an experiment.

$$\iint_D = \int_{\partial D} f\mathrm{d}z = \int_{\partial D} f\mathrm{d}x + i\int_{\partial D} f\mathrm{d}y$$

$z = x + iy$ so $\mathrm{d}z = \mathrm{d}x + i\mathrm{d}y$ Let's try this now

$$\int_{\partial D} f\mathrm{d}z = \iint_D -\frac{\partial f}{\partial y}\mathrm{d}x\mathrm{d}y + i\iint_D \frac{\partial f}{\partial x}\mathrm{d}x\mathrm{d}y$$

$$= \iint_D \left(i\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right)\mathrm{d}x\mathrm{d}y = i\iint_D \left(\frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y}\right)\mathrm{d}x\mathrm{d}y$$

If f is put to Cauchy-Riemann equation then we get,

$$0$$

If f is differentiable at every point then the cauchy riemann equation states that $\frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y} = 0$. Therefore the line integral $\int_{\partial D} f\mathrm{d}z = 0$. So the integral $\int f\mathrm{d}z$ is 0 on closed paths. Therefore it's independent of the path. Therefore f has an anti-derivative.

Riemann Cauchy equation needs a derivation by me.

Back to $f'(z)$, suppose f is differentiable at z and $f'(z) \neq 0$ then let's try to see about directional derivative f at z.

Consider z and $z + h$ that is a straight line. We are looking at $\frac{f(z+th)}{\mathrm{d}t}$ at $t = 0$.

$$\lim_{t\to 0} \frac{f(z+th) - f(z)}{t} \approx \frac{f'(z)th}{t}$$

Figure 2: Derivative of Derivative might not always exist.

So the complex projection of the derivative along h that can be thought of as a vector is a directional derivative (along θ), $= f'(z)e^{i\theta}$. So rotating the complex number by an angle θ.

Complex differentials preserve angles at points where $f'(z) \neq 0$. Lengths can vary but angles are preserved, hence conformal.