# Computational Complex Analysis : : Class 08

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## 1 Review

Suppose $f$ is differentiable, then $\overline{f(z)}$ is also differentiable.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \implies \overline{f(z)} = \sum_{n=0}^{\infty} \overline{a_n} z^n$$

Line integrals for $\mathbb{R}^n$. We deal with curve in $\mathbb{C}$, which are $C^1$ that means it has continuous derivatives (piecewise). We take line integrals of vector fields to be specific. In $\mathbb{R}^2$ we will use the integrand as a vector field although it's

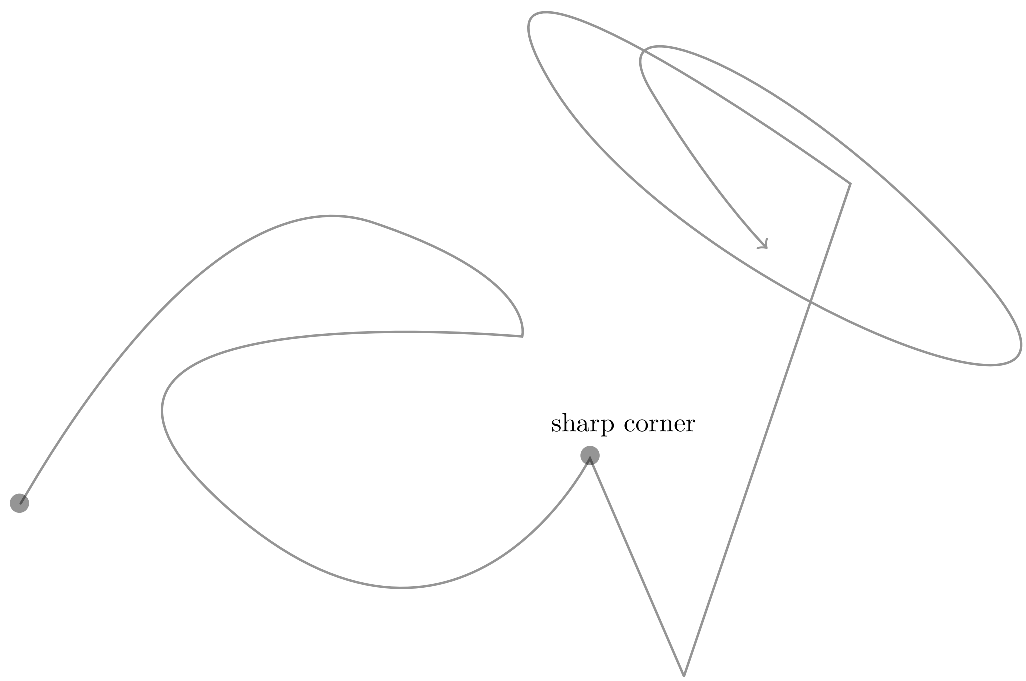


Figure 1: Paths with piece wise differentiability

not, it's a complex number $f(z)$. If curve is represented as $\gamma(t)$ and we represent it between $(a, b)$ (more correctly $a \le t \le b = [a, b]$), the line integral is

$$\int f \,\mathrm{d}t = \int_a^b f(\gamma(t))\gamma'(t)\mathrm{d}t$$

Line integrals do not depend on how $\gamma(t)$ the curve is parametrized. An example can be something going from 0 to $1 + i$,

$$\int_\gamma e^z \mathrm{d}z = \int_0^1 e^{1+i} t(1+i)\mathrm{d}t = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} e^{(1+i)t}\mathrm{d}t$$

We had $\gamma = t + it$.

How about an integral around a circle counter clockwise,

$$\int_C \mathrm{d}z \frac{1}{z}$$

Parametrize the circle by $z = re^{it}$.

$$= \int_0^{2\pi} \frac{re^{it} i \mathrm{d}t}{re^{it}}$$

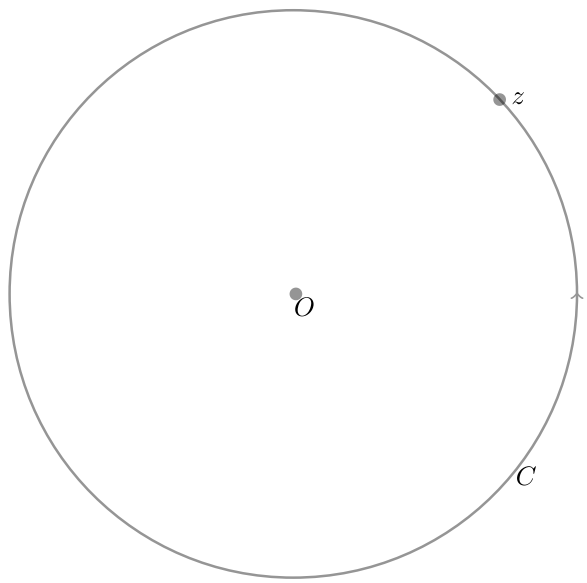


Figure 2: Taking integral around the path of a circle counterclockwise

$$\int_C \mathrm{d}z \frac{1}{z^2} = \int_0^{2\pi} \frac{ire^{it}\mathrm{d}t}{r^2 e^{2it}}$$

$$= \frac{i}{r} \int_0^{2\pi} e^{-2it} \mathrm{d}t = 2\pi \frac{i}{r} (0)$$

Turns out

$$\int z^n \mathrm{d}z = 0; n \neq -1$$

$$\int z^{-1} \mathrm{d}z = 2\pi i$$

We can do other calculations like integrating each terms one by one,

$$\int \sum_{n=-\infty}^{\infty} a_n z^n \mathrm{d}z = \sum_{n=\infty}^{\infty} a_n \int z^n \mathrm{d}z = 2\pi i a_{-1}$$

# 2 Fundamental Theorem of Calculus

Suppose $f$ is differentiable, and let's do an integral of $f$ from $a$ to $b$,

$$\int_a^b f'(z)\, \mathrm{d}z = f(b) - f(a)$$

Let $\vec{F}$ be a vector field. Then these two statements are equivalent.

1. If $\vec{F} = \nabla f$ then it's line integral over any closed loop (starts and ends at the same point) is 0.
2. Converse is also valid. Theorem in Churchill, let $f$ be a continuous function which is complex valued, defined on a connected open set in complex plane. Then the line integral

$$\int f \, \mathrm{d}z$$

on any loop equals 0 if and only if there exists a function that $F' = f$. Book calls $F$ as anti-derivative, but Prof hates it.

Proof is if $F' = f$ then

$$\int_{\text{loop}} f(z) \, \mathrm{d}z = \int_{\text{loop}} F'(z) \mathrm{d}z = F(z) - F(z_0)$$

Conversely suppose always $\int_{\text{loop}} f(z) \mathrm{d}z = 0$. Then define $F(z) = \int_{\text{path from} z_0 \text{ to } z} f(w) \, \mathrm{d}w$. This does not depend on the path.