Computational Complex Analysis : : Class 07

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Let's start with a power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

And this has $f(z_0) = a_0$. And $\sum_{n=0}^{\infty} a_n 0^n = a_0(0)^n$. A good way of writing the series is instead $a_0 + \sum_{n=1}^{\infty} a_n (z - z_0)^n$. Let's have a derivative,

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$$

$$f''(z) = \sum_{n=2}^{\infty} n(n-1) a_n (z - z_0)^{n-2}$$

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)(\cdots)(n-k+1) a_n (z - z_0)^{n-k}$$

$$f^{(k)}(z_0) = k(k-1)(\cdots)(2)(1) a_k$$

$$a_k = \frac{f^{(k)}(z_0)}{k!}$$

The taylor series of f centered at z_0 is

$$\sum_{n=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k$$

Theorem 1. Suppose $f(z)$ power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ with $R > 0$, suppose $\exists$ sequence of points $\in \mathbb{C}$ converging to z_0 so that $f() = 0$ at every one of these points. Then of course $f(z_0) = 0$ and $f = 0 \ \forall |z - z_0| < R$.

Proof. Proof by contradiction, suppose $f = 0$ is false, there will be smallest k such that $a_k \neq 0$.

$$f(z) = \sum_{k=0}^{\infty} a_{k+n} (z - z_0)^{k+n} = a_k (z - z_0)^k + \sum_{n=1}^{\infty} (a_{k+n}/a_k)(z - z_0)^{k+n}$$

$$a_k (z - z_0)^k \left[1 + \sum_{n=1}^{\infty} \frac{a_{k+n}}{a_k} (z - z_0)^n \right]$$

From continuity we cannot have a zero series shown in the third bracket. □

1 Changing Center of Convergence

$$\frac{1}{1 - z} = \sum_{n=0}^{\infty} z^n$$

$$\frac{1}{1 - z} = \frac{1}{\frac{2}{3} - z + \frac{1}{3}} = \frac{3}{1 - (3z - 2)}$$

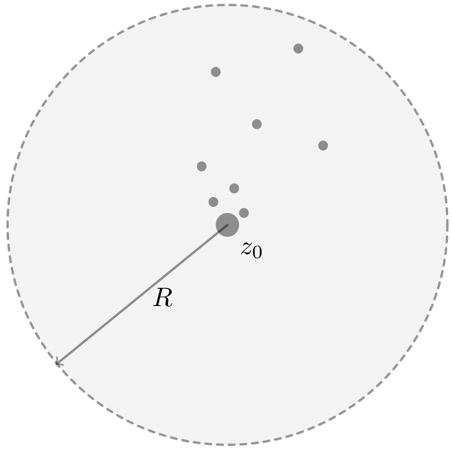

Figure 1: converging to a center point for a series

This can be written exactly like a summation,

$$= 3\sum_{n=0}^{\infty}(3z-2)^n$$

Setting the point of convergence at $-\frac{2}{3}$.

$$\frac{1}{1-z} = \frac{1}{\frac{1}{3}-(z-\frac{2}{3})}$$
$$= \frac{1}{3}\sum_{n=0}^{\infty}\left(z-\frac{2}{3}\right)^n$$

Theorem 2. Let D be an open connected subset of the complex plane $\mathbb{C}$. Suppose f is defined on D and is differentiable on every point of D. Suppose (radical assumption) the derivative is always zero for all on D. The conclusion is, f is constant.

Proof. $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ □

2 Logarithm

Consider the function

$$f(z) = \log(1-z)$$

near $z = 0$, log is differentiable but an argument is needed. If $z \to 1$, we will be near where $\log 0$ might appear. So unit disk, use principle value of $\log(1-z)$. If the first disk was centered at $z = 0$, we have another disk at $z = 1$. We can use the principle argument for $1-z$. This is differentiable for $|z| < 1$.

$$\frac{\mathrm{d}}{\mathrm{d}z}\log(1-z) = \frac{-1}{1-z} = -\sum_{n=0}^{\infty} z^n$$

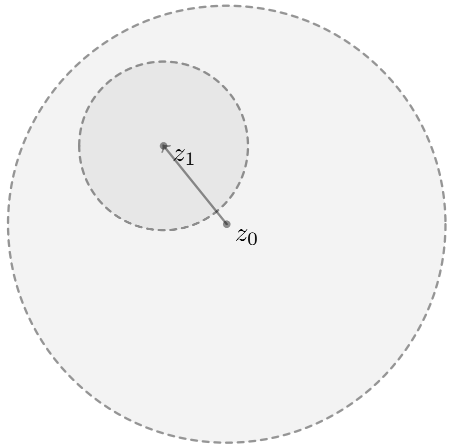

Figure 2: changing the center of convergence

Try to integrate,

$$\log(1-z) = -\sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1} + C$$

We can see $C = 0, \impliedby z = 0$.

$$\log(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n}$$

Radius of convergence of $\sum_{n=1}^{\infty} \frac{z^n}{n} = 1$. $z \to 1$ we get harmonic series that tries to blow up.

$$\ln 2 = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{4} - \cdots$$

The series and the equation are both valid if $|z| < 1$ except for $z = 1$.

2.1 Regarding next class ”Friday”

- Line Integral (212 Ref)
- Green’s Theorem
- Amazing Results (Monday), for instance Cauchy’s Integral Theorem. I need to read at home lol.

Frank Jones: “I can’t resist to say what the result is”.