# Computational Complex Analysis : : Class 04

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Ahmed Saad Sabit, Rice University

Homework handout is being analyzed.

## 1 Complex Logarithms

Let $z \neq 0$ be arbitrary, then using polar representation:

$$z = re^{i\theta}$$

Now $r = |z|$ and $\theta = \arg(z)$ where the argument can't be an actual function because it has infinitely many outputs.

Warning, never write the $\ln(z)$ where $z \in \mathbb{C}$. We will use log, $\log z = \log r + \log e^{i\theta}$. We get $\ln r + i\theta$.

Definition 1. For any $z \neq 0$ complex number, the log of $z$ is defined to be,

$$\log z = \ln |z| + i \arg(z)$$

Hence turns out, $\log z$ and $\arg z$ has the same sort of ambiguity.

$$\arg(z + 2\pi) = \arg(z)$$

$$\log(z + 2\pi) = \log(z)$$

We are computing $e^{\log z}$ which is,

$$e^{\log z} = e^{\ln |z|} e^{i \arg(z)} = |z| e^{i \arg(z)} = z$$

That is $z$ no matter of the choice of $\arg z$.

$$\begin{aligned} \log(zw) = \ln |zw| + i \arg(zw) &= \ln |z| + \ln |w| + i(\arg z + \arg w) + 2\pi i k \\ &= \log |z| + \log |w| + 2\pi i k \end{aligned}$$

## 2 Differentiation

This is where complex analysis becomes very different from real analysis.

Math 101: A function $f(t)$ is differentiable at $t_0$ if $\lim_{h \to 0} \frac{f(t_0 + h) - f(t_0)}{h}$ is $f'(t_0)$. Here is math 382,

Definition 2. A function $f = f(z_0)$ is differentiable at $z_0$ if,

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = f'(z_0)$$

Here, $h \in \mathbb{C}$ and $h \neq 0$. However this is a huge definition.

Examples, let's say $f(z) = z^2$. Then having,

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = \lim_{h \to 0} \frac{(z_0 + h)^2 - z_0^2}{h} = \lim_{h \to 0} \frac{2z_0 h + h^2}{h} = \lim_{h \to 0} 2z_0 + h = 2z_0$$

Theorem 1. If $f$ is a differentiable function at $z_0$, then $f$ is continuous at $z_0$.

**Proof.**

$$\begin{aligned} f'(z_0) &= \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \\ &= \lim_{h \to 0} \frac{1}{h} \left( f(z_0 + h) - f(z_0) \right). \end{aligned}$$

For this $\frac{1}{h}$ as it blows up, we must have the parenthesis tend to zero otherwise we don't have a limit. □

Definition 3. Product Rule.

$$(fg)' = fg' + f'g$$

Given the function $\frac{1}{f}$ is also differentiable at $z_0$.

Let's take $f(z) = Re(z)$. It's not differentiable at 0.

$$(f(h) - f(0))\frac{1}{h} = \frac{Re(h)}{h} = \frac{x}{x + iy}$$

This has no limit.

The complex input function can be written as $f(z) = f(x + iy) = f*(x, y)$.

Thus if $h \in \mathbb{R}$ then, $\frac{f*(x+h,y)}{h}$ is simply $\frac{\partial f*}{\partial x}$. For $h \in \mathbb{C} = il$ we get $\frac{f*(x,y+l)}{il}$ is giving us $\frac{\partial f*}{\partial y}$.

$$\frac{\partial f*}{\partial x} = \frac{1}{i} \frac{\partial f*}{\partial y}$$

This is the Cauchy-Riemann Equation.