Computational Complex Analsys

January 10, 2024

Ahmed Saad Sabit

Definition 1. Definitions on Hyperbolic Trigonometry

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

And we proved that this had the usual properties that solve $ke^{z+w} = e^z e^w$ webreak them into even and odd,

$$= \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$$

$$= \cosh(z) + \sinh(z)$$

We take the same thing and replace z with $-z$, and nothing happens in the first term because there is a positive power,

$$e^{-z} = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = \cosh(z) - \sinh(z)$$

Now using them we have,

$$\cosh(z) = \frac{e^z + e^{-z}}{2}$$

$$\sinh(z) = \frac{e^z - e^{-z}}{2}$$

$$\cosh(z)\sinh(z) = \frac{e^{2z} - e^{-2z}}{4} = \frac{\sinh(2z)}{2}$$

$$\sinh(2z) = 2\cosh(z)\sinh(z)$$

Calc 102 review,

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

Now we are going to put iz, hence,

$$e^{iz} = \sum_{n=0}^{\infty} \frac{i^n z^n}{n!} = \sum \frac{i^{2n}}{(2n)!} z^{2n} + \sum \frac{i^{2n+1}}{(2n+1)!} z^{2n+1}$$

From our Calc 102 review now we have,

$$e^{iz} = \cos(z) + i\sin(z)$$

Note In the book we produce the exponential functions from the $\sin z$ and $\cos z$, but here we take the reverse approach because we consider that e^{iz} is more fundamental.

This gives us,

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2}$$

Let's do a plot of $e^{i\theta}$.

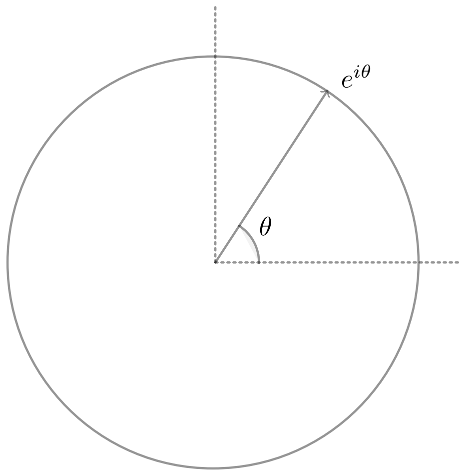

Figure 1: Plot of complex power

$$\begin{aligned} z &= x + iy \\ &= r\cos\theta + ir\sin\theta \\ &= r\left(\cos\theta + i\sin\theta\right) = re^{i\theta} \end{aligned}$$

This gives,

$$\left(r_1 e^{i\theta_1}\right)\left(r_2 e^{i\theta_2}\right) = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

$$z = |z| e^{i(\arg z)}$$

The ambiguity is that the argument can be $+2\pi$.

If we multiply all points in $\mathbb{C} without \{0\}$ by $re^{i\theta}$, then the result is, *multiply by the modulus r and add θ to the argument.* Exponentiating is rotation.

Problem 1. Page 20: Exercise 01.

$$\left(\sqrt{3} + i\right)^7$$

Solution. We can do it in one go (without doing this 7 times). We are looking at $e^{i\frac{\pi}{6}} = \frac{\sqrt{3}}{2} + i\frac{1}{2}$.

$$\left(\sqrt{3} + i\right)^7 = \left(2e^{i\frac{\pi}{6}}\right)^7 = 2^7 e^{i\frac{7\pi}{6}}$$

I see that,

$$\frac{7\pi}{6} = \frac{\pi + 6\pi}{6} = \pi + \frac{1}{6}\pi$$

So what we get is,

$$= -2^6 \left(\sqrt{3} + i\right)$$

□

Problem 2. Find all the n-th root of a complex number. w is unknown. $n \in \mathbb{Z}$.

Solution.

$$z = w^n$$

We want all possible values of w. z is given. First, we may as well assume the modulus of w is 1. So, find the n-th root of 1.
We need to find z such that, $z^n = 1$.

$$z^n = 1 = e^{2\pi i} = e^{4\pi i} = \cdots$$

So, we have,

$$z = 1, e^{2\frac{i\pi}{n}}, e^{4\frac{i\pi}{n}}, \ldots$$

Observation, consider the polynomial of degree n,

$$z^n - 1/z^{2\pi i k \frac{1}{n}} - 1$$

Illustration,

$$\frac{z^4 - 1}{z - 1}$$

These have no remainder as exact divisions.

$$z^n - 1 = \prod_{k=0}^{n-1} \left(z - e^{2\pi i \frac{k}{n}}\right)$$

Wait bro what is happening ....

$$z^4 - 1 = (z - 1)\left(z - e^{\pi \frac{i}{4}}\right)\left(z - e^{2\pi \frac{i}{4}}\right)\left(z - e^{3\pi \frac{i}{4}}\right)$$

□