Honors Linear Algebra : : Class

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Exercise 3E6

Suppose U, W are subspaces of V. One of the translates of U is a translate of U is the translate of W.

$$v + U = x + W$$

Then U and W are same to prove.

From what's given, $v = x + w$. So,

$$v - x \in W$$

Suppose $u \in U$. Then $v + u$ belongs to $v + U$. And this equals to $x + W$. So,

$$u \in x - v + W$$

Hence $U \subset W$. Likewise $W \subset U$ and $U = W$

3F: Duality

Definition 1. Consider the linear functions from V to the scalars.

$$\mathcal{L}(V, \mathbb{F})$$

Each one of these $\phi \in \mathcal{L}(V, \mathbb{F})$ is a linear function from V to the scalars. We call *Linear Functional* on V. An added notation would be V' on this space would be set of all functionals on V.

An example can be $V = \mathcal{P}(\mathbb{F})$. A functional can be

$$\int_0^1 p \, \mathrm{d}x$$

Another example can be evaluation at, say, 10,

$$p(10)$$

Functional on $\mathbb{F}^\infty$ can be

$$\vec{x} = (x_1, x_2, \ldots, x_n, \ldots)$$

A functional can be $\phi \in \mathcal{L}(\mathbb{F}^\infty, \mathbb{F})$,

$$\phi(x) = x_1 + \ldots + x_n$$

It should be apparent that V' is a vector space. Now, given for a basis V, we obtain the very important dual basis for V'

$$\phi_k(v_j) = 1 \text{ or } 0$$

Here it's one if $k = j$. These linear functionals are linearly independent. For if,

$$c_1\phi_1 + \ldots + c_m\phi_m = 0$$

Then evaluate at $v = v_k$.

$$c_k\phi_k(v_k) = 0$$

$$c_k = 0$$

From here we can say

$$\dim V' = \dim V$$

$$\vec{v} = \phi_1(\vec{v})\vec{v}_1 + \ldots + \phi_m(\vec{v})\vec{v}_m$$

Given $T \in \mathcal{L}(V, W)$,

$$V \to W$$

We define dual of T by

$$W' \to V'$$

Definition 2. T' start with a linear functional ϕ on W.

$$T'(\phi)(v) = \phi(T(v))$$

$$(T' \cdot \phi) \cdot v = \phi \cdot (T \cdot v)$$

Hence

$$T' \cdot \phi = \phi \cdot T$$

Null Space and Range of Dual of Linear Map

Definition 3. 3.121

Example 3.122

EX3F1

Each linear functional is surjective or 0. This is easy to see that if $\phi(v) \neq 0$ then $\lambda\phi(v) = \phi(\lambda v)$ so you can have any value as you want.