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Products and Quotients of Vector Spaces

Products: Suppose $V_1, \ldots, V_m$ are vector spaces (same $\mathbb{F}$) then we define their product

$$V_1 \times V_2 \times \cdots$$

The product is a cartesian product. As the set of all lists of the form,

$$(v_1, v_2, v_3, \ldots)$$

We also define the natural choices for addition and scalar multiplication.

$$(v_1, \ldots, v_m) + (w_1, \ldots, w_m) = (v_1 + w_1, \ldots)$$

$$\lambda(v_1, \ldots, v_m) = (\lambda v_1, \ldots, \lambda v_m)$$

Easy to check, this is a vector space, and these are finite dimensional,

$$\dim(v_1 \times \ldots \times v_m) = \sum_{k=1}^{m} \dim v_k$$

Basis of V_1, Basis of V_2.

Something in the way

Definition 1. Suppose U is a subspace of V. Then all possible translatables of U,

$$\vec{v} + U, \quad \vec{v} \in V$$

We just add a single vector to a whole subspace, so it translates from origin, otherwise it's mostly the same.

$$V/U = \{\vec{v} + U : \vec{v} \in V\}$$

To define addition and subtraction

$$\vec{v_1} + U + \vec{v_2} + U = (v_1) + v_2 + U$$

$$\lambda(\vec{v} + U) = \lambda\vec{v} + U$$

The quotient map

$$V \to V/U$$

Here the linear transform is called π.

$$\pi(\vec{v}) = \vec{v} + U$$

$$\pi\mathcal{L}(V, V/U)$$

$$\text{null } \pi = U$$

$$\text{range } \pi = V/U$$

$$\dim V = \dim \mathrm{null}\pi + \dim \mathrm{range}\pi = \dim U$$
$$\dim \mathrm{range}\pi = \dim V - \dim U$$

Definition 2.

$$\tilde{T} \in \mathcal{L}(V/\mathrm{null}T) \to W$$

$$\tilde{T}(v + \mathrm{null}T) = Tv$$

Exercises

$$U = \{(x, y, z) \in \mathbb{R}^3 : 2x + 3y + 5z = 0\}$$
$$A \subset \mathbb{R}^3$$

Prove that A is a translate of U if and only if there exists a real number such that A is the set of (x, y, z) such that

$$2x + 3y + 5z = C$$