Honors Linear Algebra : : Class 10

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Matrix Multiplication

$$ST(u_k) = S(Tu_k) = S(B_{1,k}v_1 + B_{2,k}v_2 + \ldots + B_{1,n}v_n)$$

These B are just scalars.

$$= B_{1,k}(A_{11}w_1 + \ldots + A_{m,1}w_m) + \ldots + B_{n,k}(A_{1n}w_1 + \ldots + A_{m,n}w_m)$$

$$= A_{1,1}B_{1,k} + \ldots + A_{1,n}B_{n,k})w_1 + \ldots + (A_{m,1}B_{1,k} + \ldots + A_{m,n}B_{n,k})w_m$$

Axler says, we define the matrix AB to fit this scheme.

Hence,

$$^{(m\times n)}AB^{(n\times p)}[= (m \times p)] \implies (AB)_{j,k} = A_{j,1}B_{1,k} + \ldots + A_{j,n}B_{n,k} = \sum_{l=1}^{n} A_{j,l}B_{l,k}$$

This sounds = „j-th row of A times k-th column of B"

$$\begin{bmatrix} A_{j,1} & \ldots & A_{j,n} \end{bmatrix} \begin{bmatrix} B_{1,k} \\ \vdots \\ B_{n,k} \end{bmatrix}$$

Conclusion,

$$\mathcal{M}(ST) = \mathcal{M}(S)\,\mathcal{M}(T)$$

Some notation, we have a matrix $A_{j,k}$

$$A_{j,\circ} \implies A \quad \text{row j}$$

$$A_{\circ,k} \implies A \quad \text{col j}$$

3.46

$$(AB)_{j,k} = A_{j,\circ}B_{\circ,k}$$

3.51 On Strange notation

Suppose C is an $m \times c$ matrix. Suppose R is a $c \times n$ matrix. Now if $1 \leq k \leq n$ then column k of CR is a linear combination of the columns of C with the coefficients coming from column k of R.

Do the same with rows.

$$(CR)_{j,k} = C_{j,l}R_{l,k} = \sum_{l}(AB)_{j,k} = A_{j,\circ} \cdot B_{\circ,k}$$

If you have a repeat of the subscript, you take a sum over it.

$$(CR)_{\circ,k} = C_{\circ,l}R_{l,k}$$

The k-th column of CR is a linear combination of the columns of C with coefficient from k-th column of R.

Rank

Axler P78, suppose A is a $m \times n$ matrix with column rank $C \geq 1$. The list of columns of A is

$$A_{\cdot,1}, \ldots, A_{\cdot,n}$$

has a span A basis for this span that can be chosen from $A_{\cdot,1}, \ldots, A_{\cdot,n}$

This basis has C vectors. Put these vectors together as columns of a new matrix C. Then for the original matrix each column $A_{\circ,k}$ is a linear combination of these. We can write $A = CR$ $A_{\circ,k}$ is a linear combination of C_k 's.

3.57 column rank is equal to row rank.

Proof.

$$A = CR$$

The rows of A (3.51(b)) are linear combination of rows of R. □

Invertible

A linear map T is $T \in \mathcal{L}(V, W)$ is invertible if

$$S : W \to V$$

such that $ST = I_v$ and $TS = I_w$ if S exists, then it's unique.