Honors Linear Algebra : : Class 09

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Linear Transformation in Matrix Form

The linear transformation of $V \to W$ can be shown by $T \in \mathcal{L}(V, W)$. Let's make individual basis

$$v_1, \ldots, v_n$$

$$w_1, \ldots, w_m$$

Then Tv_1 is a unique linear combination of $w_1, \ldots, w_m$. Define $A_{j,k}$ to be scalars.

$$Tv_k = A_{1,k}w_1 + A_{2,k}w_2 + \ldots + A_{n,k}w_n$$

Here $1 \leq k \leq m$ and we arrange these numbers

$$\begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}$$

Horizontally think about $w_1, w_2, \ldots, w_n$ and vertically think about $v_1, v_2, \ldots, v_m$. m, n are confusing, we should use sub letters that are not even in the same language,

$$A_{\aleph,\beta}$$

This matrix is known to be the linear transformation.

$$\mathcal{M}(T, (v_1, \ldots, v_n), (w_1, \ldots, w_m))$$

We can call this a matrix of T if we are not worried about the basis.

Example: Beispiel

$$T \in \mathcal{L}(\mathbb{F}^2, \mathbb{F}^3)$$

$$T(x, y) = (x + 3y, 2x + 5y, 7x + 9y)$$

Standard bases,

$$\mathbb{F}^2 : (1, 0), (0, 1)$$

$$\mathbb{F}^3 : (1, 0, 0), (0, 1, 0), (0, 0, 1)$$

The matrix would be

$$\begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 7 & 9 \end{pmatrix}$$

Example

$$T \in \mathcal{L}(\mathcal{P}^3, \mathcal{P}^2)$$

And define

$$Dp = p'$$

So standard basis

$$\mathcal{P}^3 = (1, x, x^2, x^3)$$

$$\mathcal{P}^2 = (1, x, x^2)$$

Now checking

$$D(1) = 0$$
$$D(x) = 1$$
$$D(x^2) = 2x$$
$$D(x^3) = 3x^2$$

This gives (imagine $1, x, x^2, x^3$ horizontally to be the input basis)

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Exercise 4

Some $\mathcal{P}^3 \to \mathcal{P}^2$. Basis for $\mathcal{P}^3$ is $\left(\frac{x^3}{3}, \frac{x^2}{2}, x, 1\right)$ and for $\mathcal{P}^2$ it is $(x^2, x, 1)$. The matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Suppose $S \in \mathcal{L}(V, W)$. Having the same basis,

$$Sv_k = B_{1,k}w_1 + B_{2,k}w_2 + \ldots B_{m,k}w_m$$

We have an $B_{[m,n]}$ matrix.

Plan for Thursday

We will use a transformation T and then S so that $V \to W \to U$ with each basis respective dimension being n, m, l. And we will show that the matrix for T followed by S is the product of the two matrix.

$$M(ST) = M(S)M(T)$$

This is composition

Exercise 5

We have a linear transformation from V to M. So look at the null space of V. Let's find a basis for it.

$$\operatorname{null} V : v_1, \ldots, v_k$$

Extend it to V basis,

$$\operatorname{null} V : v_1, \ldots, v_k, u_1, \ldots, u_l$$

Then $Tv_1, \ldots, Tv_k, Tu_1, \ldots, Tu_l$ will become

$$0, \ldots, 0, Tu_1, \ldots, Tu_l$$

Now we want to extend this to the basis of $0, \ldots, 0, Tu_1, \ldots, Tu_l; w_1, \ldots, w_p$ extended.

So we have chosen a basis for V and a basis for W. Let's use these bases to calculate the corresponding matrix: