Honors Linear Algebra : : Class 08

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Ahmed Saad Sabit, Rice University

Definition 1.

$$\text{null } T \subset V \to W$$

And range T belongs in W.

$$\dim V = \dim(\text{null}T) + \dim(\text{range}T)$$

Problem 28

Suppose $D \in \mathcal{L}(\mathcal{P}(R))$ and suppose $\deg(Dp) = \deg(p) - 1$ for every non-constant p. Prove that D is surjective.

It is enough to prove that each $x^m \in \text{range } D$.

Define the vector space $V = \text{span}\{x, x^2, \ldots, x^m\}$. And here $\dim V = m$. Define

$$T = D \mid_V$$

D is restricted just to V Now

$$\deg Tx = \deg Dx = 0$$

$$\deg Tx^2 = \deg Dx^2 = 1$$

$$\deg Tx^m = \deg Dx^m = m - 1$$

$$\dim(\text{range})T = m$$

$$\dim(\text{null}T) : p \in \text{null}T \implies Tp = 0$$

From here

$$\deg Dp = 0$$

Hence $p = 0$.

Bro this needs correctoin lmao.

Problem 15

Suppose there exists a linear map on V whose null space and range are both finite-dimensional. Prove that V is finite-dimensional.

Let $v_1, \ldots, v_m$ be the basis for null T.

Let $w_1, \ldots, w_n$ be the basis for range T.

w_k is $T(u_k)$ of some random vector for $u_k \in V$. Note $1 \leq k \leq n$. Now

$$v_1, \ldots, v_m, u_1, \ldots, u_n \in V$$

Let $x = c_1v_1 + \ldots + c_mv_m + d_1u_1 + \ldots + d_nu_n$.

$$Tx = d_1w_1 + \ldots + d_nw_n$$
$$---$$

Let $v \in V$ then

$$Tv = c_1w_1 + \ldots + c_nv_n$$
$$= c_1Tu_1 + \ldots c_nTu_n$$

Hence

$$v - c_1u_1 - \ldots - c_nu_n \in \text{null } T$$

It equals the linear combination of $v_1, \ldots, v_n$ and v is a linearly combination of $u_1, \ldots, u_n, v_1, \ldots, v_m$.

Problem 16

Suppose V and W are finite dimentional. Prove that there exists a linear map between the two from V to W if and only if $\dim V \leq \dim W$.

Let's start with a map. Assume we have T makes $V \to W$ exists injectively. Fundamental theorem says the dimension of V is

$$\dim V = \dim \text{null } T + \dim \text{range } T = 0 + \dim(\text{range } T) \leq \dim(W)$$

Converse solution we assume dimension of $\dim V$ is less than or equal to the dimension of $\dim W$ we need a linear map T here. The only way to create a map is the linear combination of the basis elements. Let

$$v_1, \ldots, v_m \in V \text{ be a basis}$$

$$w_1, \ldots, w_m \in w \text{ be a basis}$$

We already know that $m \leq n$. So define T the linear map of $\mathcal{L}(V, W)$

$$Tv_1 = w_1$$

$$Tv_m = w_m$$

(We took a shortcut though), actual definition of T

$$T(c_1v_1 + \ldots + c_mv_m) = c_1w_1 + \ldots + c_mw_m$$

$$T(\ldots) = 0 \implies c_1 = \ldots = c_m = 0$$

Original vector is 0. Pretty clear T is injective.

Problem 17

Complexification of a Vector Space

$$1(x, y) = (x, y)$$
$$i(x, y) = (-y, x)$$

Supposing V is a real vector space. Then the complexification. With scalar in $\mathbb{C}$ and scalar multiplication is

$$1(v_1, v_2) = (v_1, v_2)$$

$$i(v_1, v_2) = (-v_2, v_1)$$