Honors Multivariable Calculus : : Class 07

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Let $\mathcal{P} \in \mathcal{L}(V)$ and $P^2 = P$ prove that

$$V = \text{null}P \oplus \text{range}P$$

Suppose $u \in \text{null}P \cap \text{range}P$ and then

$$Pu = 0$$

$$u = Pv$$

$$0 = Pu = P^2 v = Pv = u$$

Now we need to show every vector can be written as the sum of the $\text{null}P \oplus \text{range}P$. Next $u \in V$ and $P(I - P) = 0$

$$u = Pu + (I - P)u$$

$$P(I - P)u = (P - P^2)u = 0u = 0$$

1 Problem 3B27

$\implies$ assume $S = TE$. Every vector in range S is of the form Sv. Hence $TEv = Sv$. $Sv \in \text{range}T$.

$\implies$ Assume range of S is $\subset$ range of T. Let' v_n be a basis for $\mathbb{V}$. Then Sv_k is Tw_k. I want to find E such that $S = TE$, $Sv_k = TEv_k$ Through defining E by $Ev_k = w_k$ then

$$E(\sum_{n=1}^{N} c_n v_n) = \sum_{n=1}^{N} c_n w_n$$

2 Fundamental Theorem of Linear Algebra

Given a linear function $T \in \mathcal{L}(V, W)$, and we assume V is finite dimension. Then the

$$\dim V = \dim(\text{null}T) + \dim(\text{range}T)$$

Null is in T and the range is in W.

Proof by us

- Choose a basis for $\text{null}T$ hence $\{\vec{v}_m\}$.
- Extend this basis to achieve a basis for $\mathbb{V}$ itself. $\{\vec{v}_m\}$ and $\{\vec{v}_{m+1} \dots \vec{v}_n\}$
- We assume that $n > m$.
- Look at the range of T, any vector of the form

$$\vec{v} = c_1 \vec{v}_1 + \dots + c_{m+1} \vec{v}_{m+1} + \dots$$

$$Tv = 0 + \dots + 0 + c_{m+1} T\vec{v}_{m+1} + \dots c_n T\vec{v}_n$$

Proof will be over if $T\vec{v}_{m+1}, \dots, T\vec{v}_n$ are linearly independent.

- Start with an equation of the form

$$d_{m+1}T\vec{v}_{m+1} + \ldots + d_nT\vec{v}_n = 0$$

$$T(d_{m+1}v_{m+1} + \ldots d_nv_n) = 0$$

  Then the equation in the bracket is belonged in null$T$.

3 Problem : 3B 28

Suppose D is a member of $\mathcal{L}(\mathcal{P}(\mathbb{R}))$ and suppose $\deg Dp$ is $\deg p - 1$ for non-constant polynomial p. Prove that D is surjective.

Proof Consider the subspace of $\mathcal{P}(\mathbb{R})$ to be the span of $x, x^2, \ldots, x^n$. Notice: no constant polynomial except 0. When we restrict D to this subspace, range$D$ to be equal to the span of $1, x, \ldots, x^n$.

Definition of degree

$$\deg(1) = 0$$

$$\deg(0) = -\infty$$

4 Problem 3B19