Honors Linear Algebra : : Class 06

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1.1 Problem 6

$\mathbb{P}_4(\mathbb{F})$ of a scalar field, is the subspace of all polynomials whose degree are less than 4.

$$U = \{p \in \mathbb{P}_4 : p(2) = p(5) = p(6)\}$$

It's 3 dimension because each constraint reduces the dimension. Find the basis.

Basis: 1 is the easiest. The second one is $(x-2)(x-5)(x-6)$. Then comes $(x-2)^2(x-5)(x-6)$ because for degree 4 at least one should be degree of 2. You can safely square any of the term. Check:

$$a + b(x-2)(x-5)(x-6) + c(x-2)^2(x-5)(x-6) = 0$$

If this is a basis then we should have $a = b = c = 0$. This equation is true for all x and here if $x = 2$ then $a = 0$. Now through factorization,

$$(x-2)(x-5)(x-6)[b + c(x-2)] = 0$$

This is zero for all polynomial values input x and thus $(x-2)(x-5)(x-6)$ is non-zero trivially hence $b+c(x-2) = 0$. From this we get $a = b = c = 0$.

Extend this basis for U to a basis for $\mathbb{P}_4(\mathbb{F})$. The dimension for $\mathbb{P}_4$ is 5, and thus we need 2 more polynomials for a basis.

$$x, x^2$$

Can serve as that.

1.2 Problem 7

$$U = \{p \in \mathbb{P}_4(\mathbb{F}) : \int_{-1}^{1} p \, \mathrm{d}x = 0\}$$

Find a basis: Look about odd functions so x, x^3 works for now. We need something with x^2.

$$\int_{-1}^{1} x^2 \, \mathrm{d}x = \frac{2}{3}$$

So we can include the basis $x^2 - \frac{1}{3}$ and similarly with x^4 we can include $x^4 - \frac{1}{5}$

$$x, x^3, x^2 - \frac{1}{3}, x^4 - \frac{1}{5}$$

Find subspace $W \subset \mathbb{P}_4(\mathbb{F})$ such that $U \oplus W = \mathbb{P}_4(\mathbb{F})$ We need one more because dimension is 5. $W = \mathrm{span}(1) = \mathbb{F}$

1.3 Problem 8

$v_1, \ldots, v_m$ is linearly independent in a vector space V and $w \in V$. Prove that

$$\dim \operatorname{span}(v_1 + w, \ldots, v_m + w) \geq m - 1$$

So what he does is $(v_j + w) - (v_k + w) = v_j - v_k$. Now we have to prove $v_2 - v_1, v_3 - v_1, \ldots, v_m - v_1$ is linearly independent. The proof is

$$c_2(v_2 - v_1) + c_3(v_3 - v_1) + \ldots + c_m(v_m - v_1) = 0$$

This is

$$c_2v_2 + c_3v_3 + \ldots + c_mv_m + (-c_2 - c_3 - c_4 - \ldots)v_1 = 0$$

1.4 Problem 14

We have $\dim V = 10$. V_1, V_2, V_3 are subspaces of dimension 7. Prove that $V_1 \cap V_2 \cap V_3 \neq \{0\}$.

Proof follows $\dim(V_2 + V_1) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$. As the left side of the equation is at most 10 or smaller than that, then we get $\dim(V_1 \cap V_2) \geq 4$. Now

$$\dim(V_1 \cap V_2 + V_3) = \dim(V_1 \cap V_2) + \dim(V_3) - \dim(V_1 \cap V_2 \cap V_3)$$

Turns out the dimension of the $\dim(V_1 \cap V_2 \cap V_3) \geq 1$.

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Quick Review

Let's have a map $T \in \mathbb{L}(v, w)$, and T maps V to W. We have surjective T, that means the range of T is all of W. All the vector in W comes from by means of T from V.

T is injective that means the null space of T is just the zero vectors. And also $T(v_1) = T(v_2) \implies v_1 = v_2$. Since T is linear, we can rewrite this as $T(v_1 - v_2) = 0 \implies v_1 - v_2 = 0$. This means $T(v) = 0 \implies v = 0$.

Surjective Injective doesn't necessarily require having a linear transform. x^3 is surjective and injective.

2.1 Problem 1

$b, c \in \mathbb{R}$, we have $f : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T(x, y, z) = (2x - 4y + 3z + b, 6x + cxyz)$$

Show that this is a linear map only if $b = c = 0$.

2.2 Problem 2

We have $T : \mathcal{P}(\mathbb{R}) \to \mathbb{R}^2$ by

$$Tp = \left(3p(4) + 5p'(6) + bp(1)p(2), \int_{-1}^{2} x^3 p(x)\,\mathrm{d}x + c\sin p(0)\right)$$

$b = c = 0$ because $bp(1)p(2)$ is a weird additive term, and sin is not a linear term.

2.3 Problem 4

$T \in \mathbb{L}(v, w)$ and $v_1, \ldots, v_m \in V$ such that $Tv_1, \ldots, Tv_m$ is linearly independent. Now prove $v_1, \ldots, v_m$ is linearly independent.

Step one to proving them is the combination $c_1 v_1 + \ldots + c_m v_m = 0$ and we want to prove $c_i = 0$. Well, use T of the combination so we have

$$T(c_i v_i) = 0$$

Proved!

3 3B

3.1 Problem 9

Suppose a transformation T is injective. So suppose $v_1, \ldots, v_m$ are linearly independent. Then prove $Tv_1, \ldots, Tv_m$ is also linearly independent.

Supposing

$$c_1 T v_1 + \ldots + c_m T v_m = 0$$

Remember that if system is injective then $\sum_{i=1}^{m} c_i v_i = 0$

$$T(c_i v_i) = 0$$

3.2 Problem 27

"27 is cute!" - Frank Jones.

Given $\mathcal{P} \in \mathcal{L}(v)$ such that $P^2 = P$. Prove that $V = \text{null}(P) \oplus \text{range}(P)$

If $v \in \text{null}(\mathcal{P})$ and $w \in \text{range}(\mathcal{P})$ $Pv = 0$ and $w = Pu$ so

$$Pw = P^2 u = Pu = w$$

Can $v = w$?

$$Pv = Pw \rightarrow 0 = Pw \rightarrow 0 = w$$

Conversely let $v \in V$ we want to find $x \in \text{null}(P)$ and $y \in \text{range}(P)$. Such that $v = x + y$.

$$Pv = Px + Py$$