# Honors Linear Algebra : : Class 05

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## 1 Exercises 2C

**(8)**

Suppose $v_1, v_2, \ldots, v_m$ are linearly independent. Dude please use vector notations. $\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_m$ are linearly independent. Suppose $\vec{w}$ is a vector too. Prove that

$$\dim(\operatorname{span}\{\vec{v}_1 + \vec{w}, \vec{v}_2 + \vec{w}, \vec{v}_3 + \vec{w}, \ldots, \vec{v}_m + \vec{w}\}) \geq m - 1$$

**(15)** $V_1, V_2, V_3$ subspaces. Such that

$$\dim V_1 + \dim V_2 + \dim V_3 > 2 \dim V$$

Prove that $V_1 \cap V_2 \cap V_3 \neq \{0\}$

**(19)** Prove or give a counter example.

$$\dim(V_1) + \dim(V_2) + \dim(V_3) = \dim(V_1 + V_2 + V_3) + \dim(V_1 \cap V_2) + \dim(V_2 \cap V_3) + \dim(V_1 \cap V_3) - \dim(V_1 \cap V_2 \cap V_3)$$

$\implies$ Let's take $\mathbb{R}^2$, $V_1$ : x-axis, $V_2$ : y-axis, $V_3 : x = y$. Sum of their three dimension is 3. $3 = 2 + 0 + 0 + 0 - 0$

**(20)** True version of 19.

## 2 Exercises 3A

**(16)** Suppose $V$ is a finite dimensional vector space and the $\dim V \geq 2$. Prove their exists linear operators $S, T \in \mathcal{L}(V)$ such that their product,

$$ST \neq TS$$

Example: there are two vectors $\vec{v}$ and $\vec{w}$ which are linearly independent. So there will be a basis $v, w, \ldots$. So all the vectors in $V$ have the form,

$$c_1\vec{v} + c_2\vec{w} + \cdots$$

Every vector is uniquely determined by saying what these vector coefficients are. $S(V)$ be defined such that,

$$S(\vec{v}) = \vec{v}$$

$$S(\vec{w}) = \vec{0}$$

$$T(\vec{v}) = 0$$

$$T(\vec{w}) = \vec{w}$$

Let's compute $ST$ and $TS$.

$$ST(\vec{v}) = S(T(\vec{v})) = S(0) = 0$$

$$TS(\vec{v}) = T(S(\vec{v})) = T(v) = 0$$
$$ST(\vec{w}) = S(T(\vec{w})) = S(\vec{v})$$
$$TS(\vec{w}) = T(0) = 0$$

**(11)** $V$ is a finite dimensional. $T \in \mathcal{L}(v)$. That means $T$ maps to itself. And $T$ commutes with every $S \in \mathcal{L}(v)$. $TS = ST$, then prove $T$ is a scalar multiple of the identity.

Remark: "I already knew this result for $n \times n$ matrices. And I have loved assigning it as Homework".

For all $f \in \mathcal{L}(V, \mathbb{F})$, which is not 0. I.E $\vec{v}$ such that $f(\vec{v}) \neq 0$. Proof: Use a basis for $V$ : $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ and $\vec{x} = \sum_{n=1}^{n} c_n \vec{v}_n$.

$$f(\vec{x}) = c_1$$

The set of all $\mathcal{L}(V, \mathbb{F})$ is called the dual space of $V$. Define $S \in \mathcal{L}(v)$ and $S(x) = f(x)v$ $ST = TS$,

$$ST(x) = S(T(x)) = f(T(x))V$$
$$TS(x) = T(S(x)) = T(f(x)v) = f(x)T(v)$$
$$f(x)T(v) = f(T(x))v$$
$$f(x) \neq 0$$
$$T(v) = \frac{f(T(x))}{f(x)}\ v$$

# 3 3B Null Space and Ranges

3.11 Null Space

Definition 1. Let $T \in \mathcal{L}(V, W)$. The null space of $T = \{v \in V | T(v) = 0\} = \text{null}(T)$

Fact: The null space of $T$ is a subspace of $T$. It's a subset because

$$T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w}) = 0$$

Hence $\vec{v} + \vec{w} \in \text{null}(T)$. Whatever from the subspace I put in $T$ I get 0.

$$\text{Range}(T) = \{T(v) | v \in V\}$$

Definition 2. $T$ is injective if the equation $T(\vec{v}_1) = T(\vec{v}_2) \implies \vec{v}_1 = \vec{v}_2$

Proof: $T(\vec{v}_1 - \vec{v}_2) = T(\vec{v}_1) - T(\vec{v}_2) = 0$, so

$$\therefore v_1 - v_2 \in \text{null}(T)$$

$T$ is injective if and only if $\text{null}(T)$ is $\{0\}$.

Definition 3. $T$ is surjective if $\text{range}(T) = W$.

# 4 Fundamental Theorem of Linear Algebra

$$\dim(V) = \dim(\text{null}(T)) + \dim(\text{range}(T))$$