Honors Linear Algebra : : Class 04

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1 Dimensions of a Sum

Let the subspaces V_1, V_2 be two finite dimensional vector space. We talked about,

$$V_1 + V_2$$

$$V_1 \cap V_2$$

$$\dim(V_1 + V_2) = \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2)$$

Proof. We begin the proof with a basis of the intersection $V_1 \cap V_2$, because then you can complete it. We extend it to a basis for V_1. Then for V_2. □

Problem 1. Exercises in Section 2C: (3) The vector is a polynomial of degree 4. $\mathbb{P}_4(\mathbb{F})$. The dimension is 5, $1, x, x^2, x^3, x^4$. U is the set of polynomials in $\mathbb{P}^4$ such that

$$U = \{p \in \mathbb{P}_4(\mathbb{F}) | p(6) = 0\}$$

Basis is $x - 6, (x-6)^2, (x-6)^3, (x-6)^4$.

Problem 2. Exercise (4)

$$\{p \in \mathbb{P}_4(F) | p''(6) = 0\}$$

Dimension of this space is 4. Because vector space of dimension 5 then put one constraint and that reduces dimension. Without 6 being a constraint we would have the second derivatives that can possibly equal 6

$$1, x, x^3, x^4$$

Now with the constraint

$$1, (x-6), (x-6)^3, (x-6)^4$$

Is there a choice of natural basis? "It is my own personal feeling of joy"

Problem 3. Exercise (10): Bernstein Polynomials in $\mathbb{P}_m(\mathbb{F})$

$$\mathbb{P}_k(x) = x^k (1-x)^{m-k}$$

For $\mathbb{P}_1(F)$

$$\mathbb{P}_0(x) = 1 - x$$
$$\mathbb{P}_1(x) = x.$$

For $\mathbb{P}_2(F)$

$$\mathbb{P}_0(x) = (1-x)^2$$
$$\mathbb{P}_1(x) = x(1-x)$$
$$\mathbb{P}_2(x) = x^2$$

Then, $p_0, p_1, \ldots, p_m$ form a basis for $\mathbb{P}_m$. Let's use the definition of linear independence. For $m = 4$

$$\mathbb{P}_0(x) = (1-x)^4$$
$$\mathbb{P}_1(x) = x(1-x)^3$$
$$\mathbb{P}_2(x) = x^2(1-x)^2$$
$$\mathbb{P}_3(x) = x^3(1-x)$$
$$\mathbb{P}_4(x) = x^4$$

$$c_0(1-x)^4 + c_1x(1-x)^3 + c_2x^2(1-x)^2 + c_3x^3(1-x) + c_4x^4 = 0$$

$x = 0$ then $c_0 = 0$. Factor out x then you can set $x = 0$.

$$x\left(c_1(1-x)^3 + c_2(1-x)^2 + c_3x^2(1-x) + c_4x^3\right) = 0$$

Problem 4. Exercise (13): Suppose U, W are $5-$dimensional subspace of $\mathbb{R}^9$. Then $U \cap W$ does not have $\{\phi\}$.

Problem 5. U, W are 4 dimensional subspaces of $\mathbb{C}^6$. Prove that, there are at least two vectors in the intersection such that neither is a scalar multiple of the others.

$$\dim(U \cap V) \geq 2$$

2 Linear Maps

The assumptions are $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$. U, V, W are vector spaces of $\mathbb{F}$. $3A$ section starts with the idea on vector space of linear maps.

Definition 1. A linear map from V to W is a function T, that maps

$$T : V \to W$$

With the following properties, each have linear structure with addition and scalar multiplication. You want to honor the definition of the vector space.

$$T(u+v) = T(u) + T(v)$$

$$T(\lambda u) = \lambda T(u)$$

Some mathematicians like to use the words, "Linear Transformation", or "Linear Operator". Linear maps are shown like, $\mathcal{L}(v, w)$. If $v = w$, $\mathcal{L}(v, v) = \mathcal{L}(v)$. Sabit will use $\mathbb{L}(v)$ instead. Hehe.

Differentiation is a linear map,

$$D : \mathbb{P}(F) \to \mathbb{P}(F)$$

So $(Df)(x) = f'(x)$.

Integration $T \in \mathbb{L}(\mathbb{P}(F), F)$, as

$$T(p) = \int_0^1 p(x)\mathrm{d}x$$

Backward shift V is all the sequences of the form $x = (x_1, x_2, x_3, \ldots)$. $Tx = (x_2, x_3, x_4, \ldots)$. Then there exists a unique $T \in \mathbb{L}(v, w)$ such that,

$$Tv_k = w_k$$

Operations on linear maps

$$T, S \in \mathbb{L}(V, W)$$

Then

$$(T + S)(\vec{v}) = T(\vec{v}) + S(\vec{v})$$

and

$$(\lambda T)(\vec{v}) = \lambda T(\vec{v})$$

. You can even have vector spaces of $\mathbb{L}$ linear maps.

2.1 Product Linear Maps

$T \in \mathbb{L}(U, V)$, $S \in \mathbb{L}(V, W)$

$$U -^T \to V -^S \to W$$

ST is defined to be

$$ST(\vec{u}) = S(T(\vec{u}))$$