Honors Linear Algebra : : Class 03

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1 Finite Dimensional Vector Spaces

Finitely many vectors spanning over a space makes the Vector Space Finite dimensional.

Definition 1. Spanning Set:

Theorem 1. Linear Dependence Lemma: Suppose $v_1, \ldots, v_m$ is a linearly dependent list of vectors in the vector space V. Then there exists v_k such that $v_k \in span(v_1, \ldots, v_n)$, and $span(v_1, \ldots, v_m) = span(v_1, \ldots, v_m, \text{ without having } v_k)$

Theorem 2. Then length of any linearly independent list is less than or equal to the length of any spanning list.

Proof. Let's prove it through induction. Namely, suppose linearly independent list of m vectors and spanning list of n vectors, assuming a contradiction $n < m$.

$$n = 4, m = 5$$

Linearly independent vectors $v_1, v, 2, v_3, v_4, v_5$. Spanning vectors w_1, w_2, w_3, w_4. Technique is we adjoin a vector u_1 into w_1, w_2, w_3, w_4. We can can remove w_4 and the system will still span, u_1, w_1, w_2, w_3 by Lemma. u_1, u_2, u_3, w_1 still spans. □

Definition 2. Let V be a finite dimensional vector space. A basis for V is a list which is both linearly independent and spanning.

An observation is in this case the length of the basis for V is independent of the choice of the basis. Then length is called the dimension of V. Examples of dimension: Here $\mathbb{P}_n$ is the set of polynomials of degree $\leq n$.

Vector Space	Dimension
$\mathbb{R}^n$	n
$\mathbb{C}^n$	n
$\mathbb{P}_n$	$n+1$
$V \oplus W$	$\dim(V) + \dim(W)$

Theorem 3. 2.30: Let V have a spanning set $w_1, w_2, \ldots, w_n$. This spanning set contains a basis for the vector space.

Proof. When can I not use w_1 for my linearly independent set? If it's stupid, if $w_1 = 0$, then don't use it. If $w_1 = 0$, delete it and go on. If not zero, choose it! So my first element of linearly independent set,

$$w_1$$

If w_2 is a multiple of w_1, I better not use it. $w_2 \neq aw_1$. Then we pick w_3 to not be a linear combination of

w_1, w_2 and keep going. And eventually we will get the result, which is a spanning list $w_1, w_2, \ldots, w_n$. By the process they are linearly independent. □

We can also have a reverse theorem,

Theorem 4. Every linearly independent list extends to a basis.

Theorem 5. If V is a finite dimensional vector space and U is a subspace of V, then, U is finite dimensional. Kind of crazy this needs a proof so I won't go into that - Frank Jones, 2024.

Theorem 6. V be a finite dimension. $U \subset V$. Then, there exists, another subspace such that $W \subset V$ such that $U \oplus W$ is V.

Proof. Choose a basis for U: $w_1, w_2, \ldots$. In the usual way, extend the list to get a basis for the vector space. We will have $w_1, \ldots, w_n, u_1,, w_n$ will form basis for U and u_n will form basis for W. □

Theorem 7. 2.42 V is finite dimensional. A spanning set of the right length is automatically a basis. An independent list of the correct length is also a basis.

Theorem 8. 2.42 Let V_1, V_2 be subspaces of a finite dimensional vector space. Then we can form $V_1 \cap V_2$, and we can form $V_1 + V_2$. These have dimensions,

$$\dim(V_1 + V_2) + \dim(V_1 \cap V_2) = \dim V_1 + \dim V_2$$

Proof. Axler is proper □

2 Section 1C Problem 12

Problem 1. Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspace contains the other. In that case $V_1 \cup V_2$ is simply V.

Union of three subspaces is a subspace if one of the three contains the other two.

3 Soul less problem given soul

Problem 2. Derivation of formula for $\sum_{k=1}^{n} k^2$

Solution. Start with $\sum_{k=1}^{n} k^3$ (bro!)

$$\sum_{1}^{n} k^3 = n^3 + \sum_{1}^{n-1} k^3 = n^3 + \sum_{k=1}^{n} (k-1)^3 = n^3 + \sum_{k=1}^{n} (k^3 - 3k^2 + 3k - 1)$$

The $\sum_{k=1}^{n} k^3$ cancels both side.

$$0 = n^3 + \sum_{k=1}^{n} (-3k^2 + 3k - 1)$$

Turns out

$$3\sum_{k=1}^{n} k^2 = n^3 + \sum_{k=1}^{n}(3k-1)$$

Using the idea of Arithmatic progression, we get,

$$= n^3 + \frac{3n^2+n}{2} = \frac{2n^3+3n^2+n}{2}$$

Proves,

$$\sum_{n=1}^{n} n^2 = \frac{n(2n^2+3n+1)}{6}$$

□

4 Home Reading

I have found this chapter to be a of tremendous confusion because nothing here makes sense to me. I will re-read this whole chapter from the beginning and note them down here.

Introduction to Finite Dimensional Vector Spaces

Some key points we are about to get blessed with are

- Linear Combinations of Lists of Vectors

Whoa whoa wait that's mouthful. So what I understand, that you can have a random circus of vectors

$$(\vec{s},\vec{t},\vec{u},\vec{p},\vec{i},\vec{d}),(\vec{s},\vec{p},\vec{a},\vec{c},\vec{e})$$

We have to list of random vectors up there, now they can get into combinations.

- Basis

These kids are small enough to be **independent** but big enough that their **linear combinations** fill up the entire space. Okay makes a lot of sense. You can literally have any random vector in $\mathbb{R}^3$ just from combining $(1,0,0),(0,1,0),(0,0,1)$. Let's say you a vector $(4,9,2)$. Then you need to

$$4(1,0,0)+9(0,1,0)+2(0,0,1)=(4,9,2)$$

Fair.

Span

Definition 3. Linear Combination: Let's have a list of vectors $\vec{v}_1,\vec{v}_2,\vec{v}_3,\vec{v}_4,\ldots$ in the same space V, then a possible linear combination is

$$\vec{m} = a_1\vec{v}_1 + \ldots + a_n\vec{v}_n$$

$\vec{m}$ is a linear combination of that vector list above.

For sake of example, let's make a vector list, $(1, 0), (3, 2), (4, 1)$. A linear combination of this vector can be

$$\vec{t} = 5(1, 0) + 8(3, 2) - 2(4, 1) = (5, 0) + (24, 16) - (8, 2) = (21, 14)$$

So we just showed for this list $(21, 14)$ is a valid linear combination.

Definition 4. Span: The set that contains all the possible linear combinations of a list of vectors $v_1, v_2, \ldots$ is called the span of the list. We can have any value of a_i here,

$$\text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots) = a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots : a_1, a_2, \ldots \in \mathbb{F}$$

We have to tediously put all the linear combinations of a list of vectors to get this specific set. Any linear combination of a list of vector is a member of the span. I like to think in this way. Say we have a few vectors $(1, 0), (3, 3)$. So the linear combination $5(1, 0) + 6(3, 3)$ is a member of the $\text{span}((1, 0), (3, 3))$

I am trying to get this into my mind straight that "span is the set of all linear combinations of a vector-list".

Problem 3. This is not a math problem. This is an actual goddamn problem I am suffering with. So there is a theorem *The span of a list of vectors in V is the* ***smallest*** *subspace containing all the vectors in the list.* I want a contrast, can we have a subspace that is not the smallest?

But wait, do we know what a subspace is (for real Sabit you are asking this to yourself now?).