Honors Multivariable Calculus : : Class 37

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$\vec{F}$ is a vector field on $\mathbb{R}^3$ oriented surface S. We think of $\int_S \vec{F} \cdot \mathrm{d}\vec{S}$ as sum over pieces of it is

$$(\text{value of } \vec{F}) \cdot (\text{positively directed until normal}) \cdot (\text{area of piece})$$

Example would be $\vec{F} = \langle x, y, z \rangle$ and hence

$$\int_S \vec{F} \cdot \mathrm{d}\vec{S} = \int_S \langle x, y, 0 \rangle \mathrm{d}\vec{S}$$

$$\sum \text{cosine of function and unit normal times area}$$

$$\int_S \vec{F} \cdot \mathrm{d}\vec{S} = \int_D \vec{F}(p(u,v)) \cdot \left(\frac{\frac{\partial p}{\partial u} \times \frac{\partial p}{\partial v}}{\left| \frac{\partial p}{\partial u} \times \frac{\partial p}{\partial v} \right|} \right) \left| \frac{\partial p}{\partial u} \times \frac{\partial p}{\partial v} \right|$$

You will have a plus minus depending on the orientation of the surface. Example. S to be the unit sphere centered at the origin oriented out.

$$\vec{F} = \langle 0, 0, z \rangle$$

$$\int_S \vec{F} \cdot \mathrm{d}\vec{S}$$

Parametrize ϕ, θ sphere

$$x = 1 \, \sin\phi \cos\theta$$

$$y = 1 \, \sin\phi \sin\theta$$

$$z = 1 \, \cos\phi$$

$$0 \leq \phi \leq \pi$$

$$0 \leq \theta \leq 2\pi$$

$$p(\phi, \theta) = \langle x(\phi, \theta), y(\phi, \theta), z(\phi, \theta) \rangle$$

$$\frac{\partial p}{\partial \phi} = \langle \cos\phi \cos\theta, \cos\phi \sin\theta, -\sin\phi \rangle$$

$$\frac{\partial p}{\partial \theta} = \langle -\sin\phi \sin\theta, \sin\phi, \cos\theta, 0 \rangle$$

$$\int_S \vec{F} \cdot \mathrm{d}\vec{S} = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \langle 0, 0, \cos\phi \rangle \left(\frac{\partial p}{\partial \phi} \times \frac{\partial p}{\partial \theta} \right) \mathrm{d}\phi \, \partial\theta$$