Honors Multivariable Calculus : : Class 34

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Vector path integrals

Motivation of what it is supposed to do: The ingredients are the vector field $\vec{F} \in \mathbb{R}^n$, then an oriented curve C (we have picked a direction). Integral of $\vec{F}$ along C should measure: Imagine C as a wire and we have a bead on this wire. The bead is supposed to move only along the wire, and it's constrained to move along C, bead moves along C in the direction of Integral of $\vec{F}$ along C is how much the force $\vec{F}$ help or hinder the motion.

Examples where $\vec{F}$ is constant. C is chosen to be straight. Imagine gravity and horizontal line,

$$\int_{C_1} \vec{F} = 0$$

If line is vertical then,

$$\int_{C_2} \vec{F} > 0$$

If we reverse the direction then,

$$\int_{C_3} \vec{F} < 0$$

Compare a C_4 that makes θ with vertical then, gravity helps a little less than C_2 that is directly pointing down.

We can see that the angle that the path makes with field matters.

Then length of path matters.

Strength of $\vec{F}$ matters.

$$\vec{F} \cdot \text{ (displacement along } C)$$

This is the answer if $\vec{F}$ is constant and C is straight. This is a fact.

Intuitive definition: break C into pieces and approximate each piece as straight, we can take the field at each point, and dot it with small piece displacement vector.

$$\vec{F} \cdot \mathrm{d}\vec{L}$$

Then you add this all up, a version of Riemann sum,

$$\int \vec{F} \cdot \mathrm{d}\vec{L}$$

Definition 1. If C is a piecewise C^1 parametrized curve in $\mathbb{R}^n$, paramtrized like,

$$p : [a, b] \to \mathbb{R}^n$$

then

$$\int_C \vec{F} \cdot \mathrm{d}\vec{s} = \int_a^b \vec{F}(p(t))p'(t)\mathrm{d}t$$

This is typically how it's written.

An example can be to take a swirl field $\vec{F} = \langle -y, x \rangle$ on $\mathbb{R}^2$. All points being in $(1, 0)$ and takes upper semi circle to $(-1, 0)$. Then lower $(1, 0) \to (-1, 0)$ and a line through axis.