Honors Multivariable Calculus : : Class 33

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The 212 definition of scalar surface integral. Say that is in $\mathbb{R}^3$. Let there be a function $f : \mathbb{R}^3 \to \mathbb{R}$. Then,

$$\iint_S f \,\mathrm{d}S$$

is "defined" using Riemann Sums. Break S into pieces. For each piece take a value of f on that piece. So the integral is going to be the small area times the function. And then add that up. That gives you a Riemann sum and consider the number of pieces goes to infinity.

A use of this is that integrating 1 like $\iint 1 \,\mathrm{d}S$ gives the total surface area. More generally, if δ is a per area density, then it gives the total. The average of f is

$$\frac{\iint f \,\mathrm{d}S}{\iint \mathrm{d}S}$$

Definition 1. $S \subset \mathbb{R}^3$ is a parametrized C^1 surface if there exists

$$T : D \to \mathbb{R}^3$$

and $D \subset \mathbb{R}^2$ has non empty interior such that T is C_1, the image of T is S and T is injective other than a set of content zero.

Here if S is a compact C_1 parametrized surface in $\mathbb{R}^3$ with parametrization $T : D \to \mathbb{R}^3$ then define $\iint_S f \,\mathrm{d}S$ to be just

$$\iint_D f \cdot T \text{ (change of coordinate)}$$

$$\iint_D f \cdot T \left| \frac{\partial T}{\partial u} \cdot \frac{\partial T}{\partial v} \right|$$

Let's take a

$$\{(x, y, z) : z = x^2 + y^2 \;\&\; z \leq 4\}$$

Let's set $f(x, y, z) = z$ and $\iint_S f \,\mathrm{d}S$, let's set

$$u = r \quad \text{and} \quad v = \theta$$

$$x = u \cos v$$

$$y = u \sin v$$

$$z = x^2 + y^2 = u^2$$

$$T(u, v) = (u \cos v, u \sin v, u^2)$$

$$0 \leq u \leq 2$$

$$0 \leq v \leq 2\pi$$

$$\iint f \,\mathrm{d}S = \int_{[0,2]\times[0,2\pi]} u^2 \sqrt{2u^4 + u^2} \,\mathrm{d}u \,\mathrm{d}v$$

Another type of parametrization,

$$T(u, v) = (u, v, u^2 + v^2)$$

Then this becomes,

$$\iint_D (u^2 + v^2) \left| \frac{\partial T}{\partial u} \times \frac{\partial T}{\partial v} \right| \mathrm{d}u \, \mathrm{d}v$$

Here D is a disk of radius 2.

$$\iint_D (u^2 + v^2) \sqrt{4u^2 + 4v^2 + 1} \, \mathrm{d}u \, \mathrm{d}v$$

Vector Field

Every point in space is a vector.

$$\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$$

Inputs are thought of as points and outputs are thought of as arrows. An easy example,

$$F(x, y) = \langle 1, 3 \rangle$$