Honors Multivariable Calculus : : Class 32

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1 Scalar Line Integral

Curve C in $\mathbb{R}^n$. Have some scalar function,

$$f : \mathbb{R}^n \to \mathbb{R}$$

And on the curve to somewhat distinguish from the integrals we wrote before,

$$\int_C f \mathrm{d}s$$

What is the intuitively supposed to be? Break C into pieces and on each piece take a value of f times a "bit of the domain", here the bits of the domain are the length. Add up and take a limit as number pieces goes to infinity and size of the piece goes to zero.

$$\int_C 1 \, \mathrm{d}s = \text{ arc length}$$

More generally if f is some curved density like mass or charge, then, you get total for the whole system.

$$\int_C f \, \mathrm{d}s = \text{ total}$$

Definition 1. A C in $\mathbb{R}^n$ is a smooth parametrized curve if $\exists C^1$ $p : I \to \mathbb{R}^n$ where I is an interval in R such that, we want the image of P to be C, and p being the parametric description of the curve above. We don't want to double count any part of the C, because if C is circle then we can end up over counting. So we are going to say that P is injective except possibly a content zero set.

Definition 2. $f : \mathbb{R}^n \to \mathbb{R}$ is a scalar function and C is a parametrized curve in $\mathbb{R}^n$ then $\int_C f \mathrm{d}s$ is defined to be

$$\int_a^b f(p(t)) \mid\mid p'(t) \mid\mid \mathrm{d}t$$

where $p : [a, b] \to \mathbb{R}^n$ is a C^1 parametrization of C.

An example would be, arclength of semi-circle of radius 1 we can parametrize it through,

$$p(t) = (\cos t, \sin t)$$

$0 \leq t \leq \pi$ and hence,

$$\int_c 1 \mathrm{d}s = \int_0^\pi 1 \mid\mid p'(t) \mid\mid \mathrm{d}t = \int_0^\pi \sqrt{(-\sin t)^2 + (\cos t)^2} \mathrm{d}t = \pi$$

$$\int_C x^6 \mathrm{d}s = \int_0^\pi \cos^6 t \mathrm{d}t \sqrt{(-\sin t)^2 + (\cos t)^2}$$

Definition can be extended for curve having sharp edges "piecewise C^1 "

$$q(t) = (t, \sqrt{1-t^2})$$

where $-1 \leq t \leq 1$, hence,

$$\int_C x^6 \mathrm{d}s = \int_{-1}^{1} t^6 \mid q'(t) \mid \mathrm{d}t$$

$$= \int_{-1}^{1} t^6 \sqrt{1^2 + \left(\frac{1}{2} \frac{-2t}{\sqrt{1-t^2}}\right)^2}$$

$$= \int_{-1}^{1} t^6 \sqrt{1 + \frac{t^2}{1-t^2}} \mathrm{d}t = \int_{-1}^{1} t^6 \sqrt{\frac{1}{1-t^2}} \mathrm{d}t$$