Honors Multivariable Calculus : : Class 31

March 27, 2024

Ahmed Saad Sabit, Rice University

We talked specifically about two things,

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \mathrm{d}t$$

And then the volume of the hypersphere,

$$V_n(R) = \frac{1}{n} R^n v_n$$

Here v_n is the surface area of the unit sphere at $\mathbb{R}^n$.

$\Gamma(1/2)$

So what we have discussed,

$$\Gamma(1/2) = \int_{-\infty}^\infty e^{-u^2} \mathrm{d}u$$

We can say,

$$\lim_{k\to\infty} \int_{-k}^k e^{-u^2} \mathrm{d}u$$

There is not anti-derivative without the limit. We want to instead define,

$$S(k) = \int_{-k}^k \int_{-k}^k e^{-x^2-y^2} \mathrm{d}y\mathrm{d}x$$

Why is this any better,

$$= \int_{-k}^k \int_{-k}^k e^{-x^2} e^{-y^2} \mathrm{d}y\mathrm{d}x = \int_{-k}^k e^{-x^2} \left(\int_{-k}^k e^{-y^2} \mathrm{d}y \right) \mathrm{d}x$$

$$= \left(\int_{-k}^k e^{-y^2} \mathrm{d}y \right) \left(\int -k^k e^{-x^2} \mathrm{d}x \right)$$

So it just showed,

$$\Gamma(1/2) = \sqrt{S(k)}$$

Now define,

$$C(k) = \int_{\text{disk of radius k cent. 0}} e^{-x^2-y^2} = \int_{\theta=0}^{2\pi} \int_{r=0}^k e^{-r^2} r \mathrm{d}r\mathrm{d}\theta$$

Drawing a circle of $C(k)$ and $C(\sqrt{2}\cdot k)$, we know $S(k)$ is between the two, which says,

$$C(k) < S(k) < C(k\sqrt{2})$$

Setting $u = -r^2$ now,

$$= \int_0^{2\pi} \left(e^u \left(-\frac{1}{2} \right) \right)_{r=0;\, u=-0}^{r=k;\, u=-k^2} \mathrm{d}\theta$$

$$= -\frac{1}{2} \int_0^{2\pi} \left(e^{-k^2} - 1 \right) \mathrm{d}\theta = -\pi \left(e^{-k^2} - 1 \right)$$

So, $C(k)$ goes to π and $C(k\sqrt{2})$ also goes to π so we get $S(k)$ to go to π too.

Hence, $S(k) = \pi$ and from there,

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}$$

$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{4}\sqrt{\pi}$$

$$\Gamma\left(\frac{7}{2}\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{15}{8}\sqrt{\pi}$$

Now we are interested on doing this over $\mathbb{R}^n$.

$$e^{-x_1^2-x_2^2\cdots-x_n^2}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\left(\int_{-\infty}^{\infty} e^{-x_1^2-x_2^2-x_3^2\cdots-x_n^2}\mathrm{d}x_1\right)\cdots\mathrm{d}x_n$$

$$= \left(\int_{-\infty}^{\infty} e^{-x_1^2}\mathrm{d}x_1\right)(\cdots)\left(\int_{-\infty}^{\infty} e^{-x_n^2}\mathrm{d}x_n\right)$$

To integrate over a hypersphere

$$\int_{\rho=0}^{\infty}\int_{\theta_1}\cdots\int_{\theta_2} e^{-\rho^2}\rho^{n-1}(\text{trig stuffs from jacobian})\mathrm{d}\theta\cdots\mathrm{d}\theta\mathrm{d}\rho$$

$$= \left(\int_0^{\infty} e^{-\rho^2}\rho^{n-1}\mathrm{d}\rho\right)\left(\int\int\int\int\int\int(\text{trig stuffs})\mathrm{d}\theta\right)$$

$$t = \rho^2 \quad \mathrm{d}t = 2\rho\mathrm{d}\rho \quad \mathrm{d}t\,(1/2) = \rho\mathrm{d}\rho$$

$$\int_0^{\infty} e^{-\rho^2}\rho^{n-1}\mathrm{d}\rho = \int_0^{\infty} e^{-t}\rho^{n-2}\rho\mathrm{d}\rho\mathrm{d}\rho = \int_0^{\infty} e^{-t}t^{\frac{n}{2}-1}\frac{1}{2}\mathrm{d}t = \frac{1}{2}\Gamma\left(\frac{n}{2}\right)$$

From here we come to conclude,

$$\sqrt{\pi}^n = \frac{1}{2}\Gamma(n/2)v_n$$

Hence we get,

$$v_n = \frac{\sqrt{\pi}^n}{\frac{1}{2}\Gamma\left(\frac{n}{2}\right)}$$

$$V_n(R) = \frac{R^n}{n}\frac{\sqrt{\pi}^n}{\frac{1}{2}\Gamma(\frac{n}{2})}$$

$$\boxed{V_n(R) = \frac{R^n\left(\sqrt{\pi}^n\right)}{\Gamma(\frac{n}{2}+1)}}$$

Computing this,

$$V_4(R) = \frac{\pi^2}{2}R^4$$

$$V_5(R) = \frac{8\pi^2}{15}R^5$$