Honors Multivariable Calculus : : Class 30 (whaatttt)

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Ahmed Saad Sabit, Rice University

We want to talk about spheres. We will soon be talking about n-dimensions. This is what 212 people are worried about.

Spherical Coordinates in $\mathbb{R}^3$

We can use more angles to coordinize. Take,

$$\rho, \phi, \theta$$

- ρ measures distance from origin, so $\rho = \sqrt{x^2 + y^2 + z^2}$.
- ϕ measures angle between $\vec{z}$ and the $\vec{r}$.
- θ measures $\vec{r}_{xy}$ with $\vec{x}$ axis.

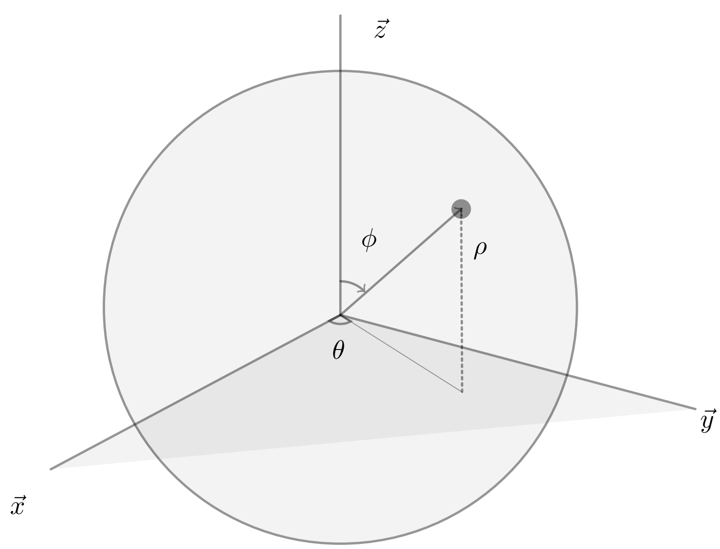

Figure 1: Spherical Coordinates (Class 30)

A cone can be described by simply,

$$z = \sqrt{x^2 + y^2}$$

$$\phi = \frac{\pi}{4}$$

The transformation from Cartesian to Polar is,

$$z = \rho \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$x = \rho \sin\phi \cos\theta$$

$$(x, y, z) = T(\rho, \phi, \theta)$$

Then,

$$\det \mathrm{d}T = \rho^k (\text{trig stuffs})$$

What can k be? $k = 2$.

$$\int_R f = \int_R' f \cdot T |\det \mathrm{d}T| \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\theta$$

The volume of R radius ball is covered in the bounds,

$$0 \le \rho \le R$$

And the angle dimensions are,

$$0 \le \phi \le \pi$$
$$0 \le \theta \le 2\pi$$

So the integral is to find the volume is,

$$\int_0^R \int_0^\pi \int_0^{2\pi} \rho^2 \sin\phi \, \mathrm{d}\theta \, \mathrm{d}\phi \, \mathrm{d}\rho$$

These are linearly independent things we are integrating so,

$$\int_0^R \rho^2 \ (\text{trig stuffs}) \ \mathrm{d}\rho$$

The (trig stuffs) is basically surface area of a unit sphere.

$$\int_0^R 4\pi\rho^2 \mathrm{d}\rho$$
$$= \frac{4\pi}{3} R^3$$

Let's work on this on $\mathbb{R}^{4+}$

$$\rho = \text{ dist from origin}$$
$$\rho^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$$

We want to come up with coordinates for the rest of these things. So,

$$\rho^2 = x_4^2 + \left(\sqrt{x_1^2 + x_2^2 + x_3^2}\right)^2 = x_4^2 + \rho^2 \sin^2\theta_1 = \rho^2 \cos^2\theta_1 + \rho^2 \sin^2\theta_1$$

We are basically saying,

$$x_4 = \rho \cos\theta$$
$$x_1^2 + x_2^2 + x_3^2 = (\rho \sin\theta_1)^2$$

Use spherical coordinates from $\mathbb{R}^3$ to deal with these, with the $\mathbb{R}^3$ case replaced by $\rho \sin\theta_1$,

$$x_3 = (\rho \sin\theta_1)\cos\theta_2$$

$$x_2 = (\rho \sin\theta_1)\sin\theta_2\cos\theta_3$$

$$x_3 = (\rho \sin\theta_1)\sin\theta_2\sin\theta_3$$

Here the bounds for the sphere are,

$$0 \leq \theta_1, \theta_2 \leq \pi$$

$$0 \leq \theta_3 \leq 2\pi$$

The volume is going to be,

$$\int_0^R \left(\int_{\theta_1} \int_{\theta_2} \int_{\theta_3} \cdots \int_{\theta_n} \right) \det \mathrm{d}T \, \mathrm{d}\theta_1 \, \mathrm{d}\theta_2 \, \mathrm{d}\theta_3 \cdots \mathrm{d}\theta_n \mathrm{d}\rho$$

Notice that,

$$\det \mathrm{d}T = \rho^{n-1}(\text{trig stuffs})$$

Hence the volume integral,

$$\int_0^R \rho^{n-1} \text{ (trig integral) } \mathrm{d}\rho = \frac{1}{n} R^n v_n$$

Find the volume in n-dimension

spoiler link here.

Definition 1. Γ function is,

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, \mathrm{d}t$$

Only defined for $z > 0$.

Problem 1.

$$\Gamma(5) = \int_0^\infty t^4 e^{-t} \mathrm{d}t = -\left[t^4 e^{-t}\right]_0^\infty + \int_0^\infty 4t^3 e^{-t} \mathrm{d}t = 4\Gamma(4)$$

We can just do this generally and as we found in Computational Complex Analysis,

$$\Gamma(n+1) = n\Gamma(n)$$

$$\Gamma(5) = 4\Gamma(4) = 4\Gamma(4) = 4 \cdot 3\Gamma(3) = 4 \cdot 3 \cdot 2\Gamma(2) = (4 \cdot 3 \cdot 2 \cdot 1)\,\Gamma(1) = (5-1)!$$

Problem 2. Another example can be shown that,

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-1/2} e^{-t} \mathrm{d}t$$

Setting,

$$u = \sqrt{t} \quad \mathrm{d}u = \frac{1}{2\sqrt{t}} \mathrm{d}t$$

So,

$$\int_0^\infty e^{-u^2} 2 \mathrm{d}u = \int_{-\infty}^\infty e^{-u^2} \mathrm{d}u$$