Honors Multivariable Calculus : : Class 29

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Say f is integrable on some compact domain $D \subset \mathbb{R}^n$. So T is going to take D' to D, one domain to another. Hence, $D' \subset \mathbb{R}^n$, and if

$$T : D' \to D$$

is C^1, the function is "onto". And T is into except -¿ ¡fudge this?¿

Example.

D a annulus ring. Inner circle 1 radius, outer is 3. We don't want to integrate using x, y but we do want to use r, θ here. So,

$$T(r, \theta) = r\cos\theta, r\sin\theta$$

So, $x = r\cos\theta$ and $y = r\sin\theta$. What is our D' in the r, θ universe? The r, θ graph lower horizontal line and upper horizontal line is contant zero. So although not injective we don't care. $r = 1$ to $r = 3$ and $\theta = 0$ to $\theta = 2\pi$. But, $\theta = 2\pi = 0$, but this is content zero so we don't quite care.

¡fudge¿ T is into (injective) except at possibly content 0 portion of D'.

Then,

$$\int_D f = \int_{D'} f \cdot T |\det T|$$

If we are trying this,

$$\int_D x^2 = \int_D' (r\cos\theta)^2 \cdot r (= |\det T|) = \int_0^{2\pi} \int_{r=1}^{r=3} r^3 \cos^2\theta \, \mathrm{d}r \, \mathrm{d}\theta$$

Average distance of all points in a circle,

$$\frac{1}{\text{area } D} \int_D \sqrt{x^2 + y^2} = \frac{1}{36\pi} \int_D \sqrt{x^2 + y^2}$$

After change of coordinates, $\sqrt{x^2 + y^2} = r$

$$\frac{1}{36\pi} \int_{\theta=0}^{2\pi} \int_{r=0}^{6} r |\det \mathrm{d}T| \, \mathrm{d}r \, \mathrm{d}\theta = \frac{1}{36\pi} \int_{\theta=0}^{2\pi} \int_{r=0}^{6} r \cdot r \, \mathrm{d}r \, \mathrm{d}\theta$$

Why do we want to change coordinates?

- Turns the problem boundary relatively easier to do.
- Makes the function (integrand) easier to deal with.
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Let's try,

$$\int_D (x - y)^3 (x^2 - y^2)^{\frac{1}{3}}$$

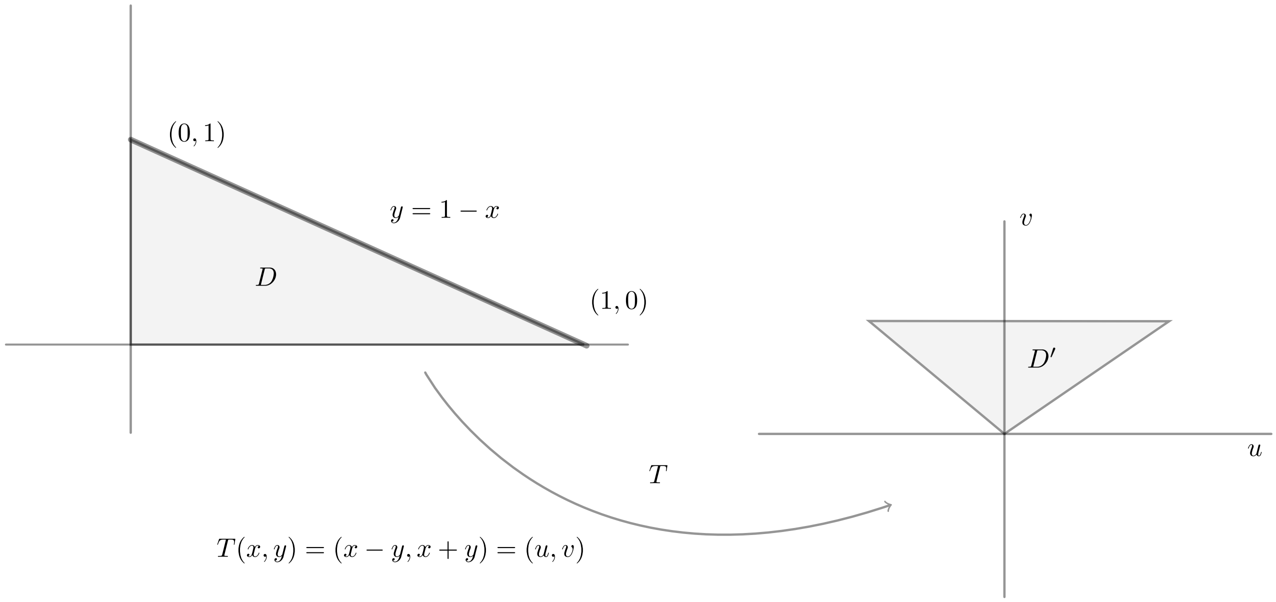

Figure 1: The region for the integration (Class 29)

Changing coordinates,

$$T(x,y) = (x-y, x+y)$$
$$x = \frac{u+v}{2} \qquad y = \frac{v-u}{2}$$
$$\frac{\partial\,(u,v)}{\partial\,(x,y)} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$
$$\det \frac{\partial\,(u,v)}{\partial\,(x,y)} = \det \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = 2$$
$$S(u,v) = (x,y)$$

We can just take inverse of the determinant of the $\frac{\partial(u,v)}{\partial(x,y)}$ for the integral.

$$\det \frac{\partial(x,y)}{\partial(u,v)} = \det \frac{\partial(u,v)}{\partial(x,y)}^{-1} = \frac{1}{2}$$

The integral is now,

$$\int_{D'} u^{\frac{10}{3}} v^{\frac{1}{3}} \frac{1}{2} \,\mathrm{d}u\,\mathrm{d}v = \int_0^1 \int_{-v}^{v} \frac{1}{2} u^{\frac{10}{3}} v^{1/3} \,\mathrm{d}u\,\mathrm{d}v$$

$$\int_{-\infty}^{\infty} e^{-x^2} \,\mathrm{d}x = \sqrt{\pi}$$