Honors Multivariable Calculus : : Class 27

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Continuous function inside a compact set is uniformly continuous.

Theorem 1. If f is continuous on a box

$$B = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$$

then setting $S(x, y) = \int_{a_3}^{b_3} f(x, y, z)\mathrm{d}z$ we have

$$\int_B f = \int_{[a_1,b_1]\times[a_2,b_2]} S(x, y)$$

This is called Fubini's Theorem. Sketch of a proof. First thing is we need to say S is integrable $(S(x, y))$. We will show that S is continuous then it will be integrable on a rectangle. Why is S continuous? We say, $(x_1, y_1) = (x + \delta x, y + \delta y)$ here $\delta x, \delta y$ are small. We want to compare $f(x, y, z)$ to $f(x_1, y_1, z)$. Let's claim these functions are close. This is required to be universal closeness. We need "uniform continuity" of f on B if this distance between $\delta x, \delta y$ are small enough then $f(x_1, y_1, z)$ and $f(x, y, z)$ are close within some ϵ'. So we have $f(x_1, y_1, z) - \epsilon' < f(x_1, y_1, z) < f(x, y, z) + \epsilon'$ for all z. Thus here the

$$\int_{a_3}^{b_3} f(x_1 y_1, z) - \epsilon' \mathrm{d}z < \int f(x_1, y_1, z)\mathrm{d}z < \int f(x_1, y_1, z) + \epsilon'$$

So we have

$$S(x, y) - \epsilon'(b_3 - a_3) < S(x_1, y_1) < S(x_1, y) + \epsilon'(b_3, -a_3)$$

So S is continuous hence integrable. From there, now. To prove.

So if division of the the "mesh" is fine enough, then $\int_{B'} S(x, y)$ is close to any Riemann Sum

$$\sum_n s(x_n, y_n)\Delta x \Delta y = \sum \left(\int f dz \right) \Delta x \Delta y$$

Remember

$$\int_B f \text{ is close to } \sum_i \left(\sum_j f(x_i, y_i, z_j)\Delta x \Delta y \Delta z \right) = \sum_i \left(\sum_j f(x_i, y_i, z_j)\Delta z \right) \Delta x \Delta y$$

We have a similarity to the equation we had built,

$$\sum_n s(x_n, y_n)\Delta x \Delta y = \sum \left(\int f dz \right) \Delta x \Delta y$$

Fixed the notation to be exact what professor wrote (couldn't type properly in class).

$$\sum_i s(x_i, y_i)\Delta x \Delta y = \sum_i \left(\int_{a_3}^{b_3} f(x_i, y_i, z) dz \right) \Delta x \Delta y$$

Here's an idea

$$\int_{a_3}^{b_3} f(x_i, y_i, z)\,\mathrm{d}z = \sum_j \left(\int_{s_{j-1}}^{s_j} f(x_i, y_i, z)\,\mathrm{d}z \right)$$

There is a mean value theorem for integration,

$$\int_a^b g(x)\mathrm{d}x = (b-a) \text{ arg value of } g \text{ on } [a,b] = (b-a)g(c)$$

for some c and g is continuous.

Not in a box? Then

$$\int_R f = \int_B f \cdot \xi_R$$

Here $\xi = 1$ if $\vec{x} \in R$ and $\xi = 0$ if not.