MATH382 note starts after MATH232
Didn't take on computer because I
felt too tired today.

MATH232
CLASS 26

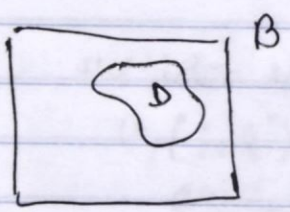
Integrating on non boxes

D bounded region in $\mathbb{R}^n$
$f : D \to \mathbb{R}$ bdd function.
$\int_D f$ defined

$\int_B f_1$ where, B = box containing D
& $f_1 = f$ on D
$= 0$ on $B \setminus D$

$f_1 = f \cdot \chi_D$

$\chi_D(\vec{x}) = \begin{cases} 1 & \vec{x} \in D \\ 0 & \text{else} \end{cases}$



If f is continuous on D
Then f_1 will be cont on B
except at ∂D.

Fact: most "nice D" is will have ∂D content zero.

EX: Graph of any cont function on cpt set is content zero.
Pf: HW

proposition: If $D \subset \mathbb{R}^n$ (compact) has boundary ∂D which is content zero, then if f is cont on D, f is integrable on D.

f integrable on a box $\forall \varepsilon > 0$, $\exists$ partition P
s.t. $U(f,P) - L(f,P) < \varepsilon$

<u>prop</u>: let B = box, f bdd on B then the integral of $\int_B f = I$ iff $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall$ partition P of B with diameter of piece (or sizes) as $< \delta$.

$U(f,P)$ and $L(f,P)$ are within ε of I.

equivalently any Riemann Sum for f on P is within ε of I.

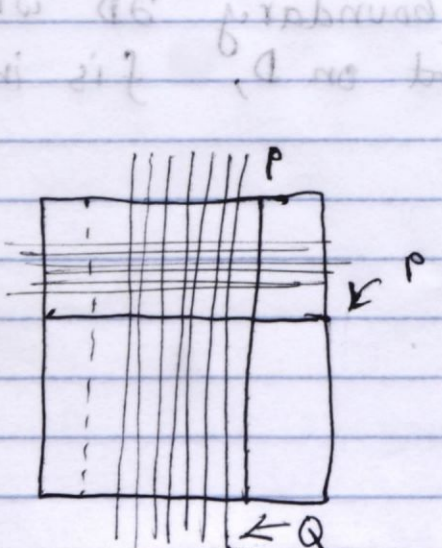
$$\int_B = \lim_{\substack{\text{size} \\ \text{supart} \\ \to 0}} \begin{pmatrix} \text{Riemann} \\ \text{sum of} \\ f \text{ on } B \end{pmatrix}$$

BROAD IDEA OF HOW YOU PROVE THIS:

Suppose f is integrable.
let $\varepsilon > 0$
how do we construct δ.

we know there is some partition. pick a fixed P such that Upper − Lower $< \varepsilon/3$.

if this case,
any refinement on P, $U - L < \varepsilon/3$



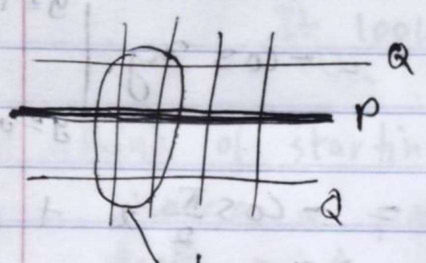
if Q is refinement of P we already are in condition.
we want to do this for all δ.

Goal to show $U(F,Q) - L(F,Q) < \varepsilon$
if Q's diameter is small enough.

Let R = refinement of Q & P

$U(f,R) - L(f,R) < \varepsilon/3$

on Q we apply P number of slices where we understand P is exactly.



boxes are getting disturbed.

the idea ~~if~~ is ~~if~~ use Q's smallest to make sure
$U(f,Q)$ is close to $U(f,R)$
and $L(f,Q)$ to $L(f,R)$

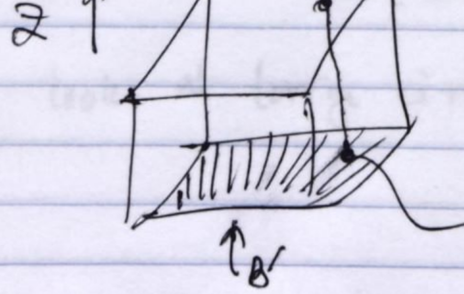
then $U(f,Q) - L(f,Q) < \varepsilon$

$\approx U(F,R) - L(f,R)$

<u>HOW TO COMPUTE INTEGRAL</u>

f is cont on $B = [a_1,b_1] \times [a_2,b_2] \times [a_3,b_3]$ in $\mathbb{R}^3$
box

define $S(x,y) = \int_{z=a_3}^{b_3} f(x,y,z)\, dz$

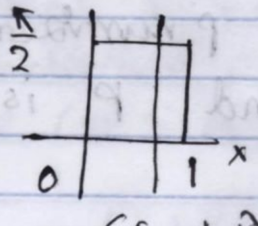


any fixed x,y, then $f(x,y,z) \ni f(z)$

~~thm~~ Theorem : $\int_B f = \int_{B'} S(x,y)$

Ex

↳ in $\mathbb{R}^2$ integrate $x\sin(xy)$ on $[0,1]\times[0,\pi/2]$



(fixed x)

$$S(x) = \int_{y=0}^{y=\pi/2} x \sin xy \, dy$$

$$= -\cos xy \Big|_{y=0}^{y=\pi/2}$$

$$S(x) = -\cos\frac{\pi}{2}x + 1$$

each strip

$$\int_0^1 S(x)\,dx = \int_0^1 \left(-\cos\frac{\pi}{2}x+1\right)dx$$

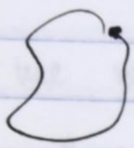
$$= \left[\frac{-\sin\frac{\pi}{2}x}{\pi/2} + x\right]_0^1$$

written as

$$\int_0^1\int_0^{\pi/2} x\sin xy\,dy\,dx$$

iterated integral

The counting priciple



start f defined inside and on the curve.

Holomorphic f never 0 on C.

$$\frac{1}{2\pi i}\int \frac{f'(z)}{f(z)}\,dz = \text{the number of zero of } f \text{ inside the curve.}$$

Another interpretation of the integral

It looks like $\int_C \left(\frac{d}{dz}\log f(z)\right) dz$

Think of starting at 0 point of the curve: choose a particular value of $\arg f(z)$ at that point.

We define a continuous determination of $\arg(z)$ along the curve, until we return to the start. There perhaps will ~~recur~~ result in a different determination of $\arg z$.

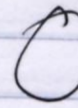
This difference has to be integer multiple of 2π.

The Argument Principle:

The change in measurement of $f(z)$ along a positive direction along the curve equals $2\pi N$, $N=\#$ zeroes in C

Let $F(z)$ be a polynomial

$$az^n + bz^{n-1} + \dots +$$

look at large circle,  on this circle, $|z| = R$

$f(z) = az^n(1 + \text{small stuff } f)$ for large $|z|$

f has exactly n zeroes in $\mathbb{C}$.

$f(z) = z^8 + az^3 + bz + c$ $\quad a, b, c \in \mathbb{R}$.

how many zeroes does f have on the first quadrant?

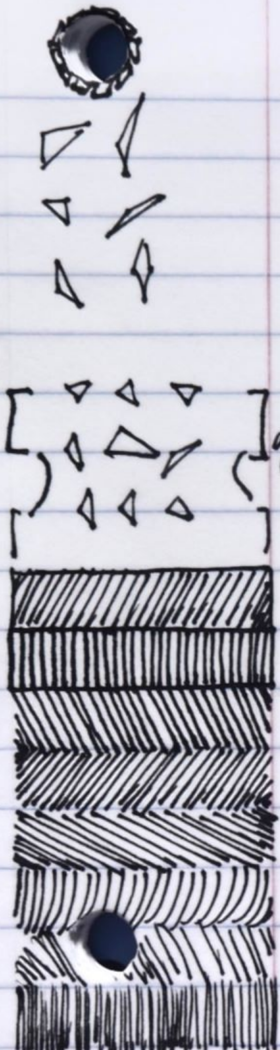
Find # zeroes of f for $\text{Real}(z) > 0$

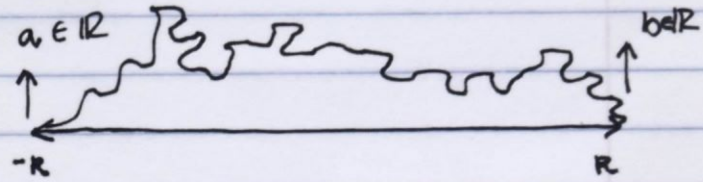
$f(iy) = (y^8 + c) + i(-ay^3 + by)$

$$\tan \arg(f(iy)) = \frac{-ay^3 + by}{y^8 + c}$$

suppose $f(z)$ is holomorphic for all $z \in \mathbb{C}$
and strangely $f(z) \in \mathbb{R}$ iff $z \in \mathbb{R}$.

we want to prove that f has a zero.

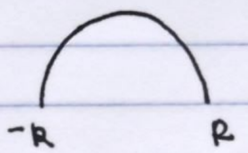


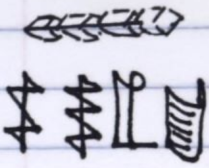


$\Downarrow$

google says $f(z) = az + b$
$a, b \in \mathbb{R}$

$$\Gamma(a) = \int_0^\infty t^{a-1} e^{-t}\, dt$$





$\Gamma(1) =$

$$\Gamma(a+1) = \int_0^\infty t^a e^{-t}\, dt = \int_0^\infty -t^a \, d(e^{-t})$$

$$= t^a(e^{-t})\Big|_0^\infty + \int_0^\infty (t^a)' e^{-t}\, dt$$

$$= \int_0^\infty a t^{a-1} e^{-t}\, dt$$

$$= a \int_0^\infty t^{a-1} e^{-t}\, dt = a\Gamma(a)$$