Honors Multivariable Calculus : : Class 25

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For all a and for all b there exist a c is a (statement S involving a, b, c). For all b there exists c such that for all a Which statement might be easier to prove here? Statement 2 is harder to prove.

The difficulty in $\epsilon - \delta$ is choosing the δ. What is the statement f is continuous on D of the form? Most likely statement 1. That's for because $\forall x$ and $\forall \epsilon > 0$ there exists $\exists \delta > 0$ such that - something.

Definition 1. f is uniformly continuous on D if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\forall x \in D$ if $|\vec{y} - \vec{x}| < \delta$ then $|f(\vec{y}) - f(\vec{x})| < \epsilon$.

An example can be $f(x) = 4x$ to be uniform continuous.

Proof. Let $\epsilon > 0$ and choose $\delta = \frac{\epsilon}{4}$, then if $x, y \in \mathbb{R}$ with $|y - x| < \delta = \frac{\epsilon}{4}$, we have,

$$|f(y) - f(x)| = |4y - 4x| = 4|y - x| < \epsilon$$

□

Another example, $f(x) = x^2$ is not a uniformly continuous on $\mathbb{R}$. Note about the Δx for Δf, moving to the left, we need narrower and narrower tolerance around x that Δx gets smaller. So there is not a single δ that can be universally okay.

Proof. We want to find one counter example which will totally negate the statement.
Choose $\epsilon = 1$. Then we are trying to claim that there is no δ that works. So for any $\delta > 0$ we can find an x and choose it such that x is greater than $\frac{1}{\delta}$. Then what happens when we take $y = x + \frac{\delta}{2}$ then we get

$$f(y) - f(x) = \left(x + \frac{\delta}{2}\right)^2 - x^2 = \delta x + \frac{\delta^2}{4} > \delta x > \epsilon$$

□

Theorem 1. If D is compact and f is continuous on D then f is a uniformly continuous on D. This is a happy fact.

Proof. Analysis □

Theorem 2. Proposition: If f is continuous on a box D then f is integrable on D. Here $f : D \to \mathbb{R}$ and $D \in \mathbb{R}^n$.

Proof. Let $\epsilon > 0$. Define $\epsilon' = \frac{\epsilon}{\text{vol of } D}$. Uniform continuity of f on D means there exists an $\delta > 0$ that $\vec{y}, \vec{x} \in D$ with $|\vec{y} - \vec{x}| < \delta$ then $|f(\vec{y}) - f(\vec{x})| < \epsilon'$. Pick a partition P such that $\vec{x}, \vec{y}$ are in the same piece P then $|\vec{y} - \vec{x}| < \delta$.
So on each piece the max of f subtracted from *min* value of f :

$$U(f, P) - L(f, P) < \sum_{\text{pieces}} (\text{vol of piece})(\text{max value of f on piece - min value of f on piece}) <$$

$$< \sum_{\text{pieces}} (\text{vol of piece})\epsilon' = \text{vol}(D)\epsilon' = e$$

□

Before we move to non-boxes, what about f is non continuous?

Definition 2. A set $X \subset \mathbb{R}^n$ has content 0 or content zero if $\forall \epsilon > 0$ $\exists$finitely many boxes $B_1, \ldots, B_k$ such that $x \subset \cup B_i$ and

$$\sum_{i=1}^{k} \text{vol}(B_i) < \varepsilon$$

$\subset \cup \cap \in$

Theorem 3. Proposition: If the set of the discontinuities of f on the box D is content zero, then f is integrable on D. When we are trying to integrate functions it's important to remember that our functions are bounded.

Proof. D and we are not assuming f is continuous. In the box D imagine some line where X is the set of discontinuities. Choose P partition such that the pieces of P that intersect X that have total volumen < (fill in the blank later). (by $X's$ content zero.)

f is uniformly continuous outside of those boxes, choose P also such that if $\vec{y}, \vec{x}$ are in a single piece of outside of these boxes then $|f(\vec{y}) - f(\vec{x})| <$ (fill in box).

Then

$$U(f, P) - L(f, P) = \sum_{n=1}^{\infty} (\text{vol of piece})(\text{min - max})$$

$$= \sum_{\text{piece that contain X}} (\text{vol})\ |\text{min - max}| + \sum_{\text{others}} (\text{vol})|\text{min - max}|$$

Now the boxes around the discontinuous part can be taken really small though the min - max would not be ssmall.

$$< \sum_{\text{pieces containing } X} (\text{vol})(\text{overall max - over min of f on D}) + \sum_{\text{other pieces}} (\text{vol})\ \epsilon'$$

$$< (\text{overall max - overall min})\ \epsilon'' + (\text{vol} D)e' < \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

□