Honors Multivariable Calculus : : Class 23

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INTEGRATION: (lesgoo)

Math 212 understanding,

$$D \subset \mathbb{R}^n$$

and let's be in $f : D \to \mathbb{R}$. The integral

$$\int_D f =$$

$$\int_{[1,5]} 3x^2 \mathrm{d}x \text{ is defined via Riemann Sums}$$

We just going to think we are summing the area of a bunch of rectangles. So think about breaking the system into Pieces.

$\int_D f$ is break D into pieces and each piece take "volume of the piece in n-dim" and multiply "value of f on piece". This is the informal but useful understanding of what the integration is doing. What does integral measure?

signed area, which is the volume of between the graph of f in $\mathbb{R}^{n+1}$ and domain $D \times \{0\} \subset \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$

If f is density like stuffs, the integral gives the mass. Defining δ density r position, moment of inertia

$$\int_D \delta r^2 = \text{ Moment of Inertia}$$

Integrals might give average. Take N values of f each $\frac{1}{N}$ of D.

$$\frac{\sum_{n=1}^{N} f(x_i)}{N} = \frac{\sum_{n=1}^{N} f(x_i) \frac{\text{area of } D}{N}}{\text{area of } D} \approx \int_D f$$

So $\int_D f$ are n dimensional volume of D, gives average of f.

Usage of Inverse Functions using Implicit Theorem

Y'all know about Polar Coordinates.

$$(x, y) = (r\cos\theta, r\sin\theta) = F(r, \theta)$$

Hence, $F : \mathbb{R}^2 \to \mathbb{R}^2$.

Definition 1. Given C^1 $\mathbb{F} : \mathbb{R}^n \to \mathbb{R}^n$ with $\vec{a} \in \mathbb{R}^n$ where $dF_{\vec{a}}$ is invertible. Then F is locally a bijection, or, there are open sets U where $a \in U$ such that $f(\vec{a}) \in V$ such that F restricted to U is a bijection between U

and V. The inverse G sending V to U is C^1, and

$$dG_{f(\vec{a})} = (dF_{\vec{a}})^{-1}$$

Proof. Let $F = (f_1, \dots, f_n)$ and $f_i : \mathbb{R}^n \to \mathbb{R}$. Consider $\mathbb{R}^{2n}$ consists of $x_1, \dots, x_n$ and $y_1, \dots, y_n$. And we got

$$y_1 = f_1(x_1, \dots, x_n)$$

$$y_n = f_n(x_1, \dots, x_n)$$

These functions show the graph of f_n. Apply implicit function theorem,

$$y_1 - f_1(x_1, \dots, x_n) = 0$$

$$g_n(x_1, \dots, x_n, y_1, \dots, y_n) = y_n - f_n(x_1, \dots, x_n) = 0$$

To apply Implicit function theorem, we need mx of

$$\frac{\partial g_i}{\partial x_j}$$

to be invertible. This is the same as mx of $-\frac{\partial f_i}{\partial x_j}$ at $\vec{a}$. This is $-dF_{\vec{a}}$. □