Honors Multivariable Calculus : : Class 22

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Definition 1. Given m functions $f_1, \dots, f_m : \mathbb{R}^n \to \mathbb{R}$ where

$$F = (f_1, \dots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$$

$\vec{a} \in \mathbb{R}^n$, where $F(\vec{a}) = \vec{c}$. Relabel $x_{n-m+1} \dots x_n$ as $z_1, \dots, z_m$. Then near $\vec{a}$, the constraint

$$F(\vec{x}) = \vec{c}$$

defines $z_1, \dots, z_m$ as implicit functions of $x_1, \dots, x_{n-m}$ if

$$\begin{pmatrix} \frac{\partial f_1}{\partial z_1} & \frac{\partial f_2}{\partial z_1} & \cdots \\ \vdots & & \\ & & \frac{\partial f_a}{\partial z_b} \end{pmatrix}$$

Example

Intersection of $x^2 + y^2 + z^2 = 3$ and $x + 2y + 3z = 6$ near $\vec{a} = (1, 1, 1)$. Can we get y, z as implicit function of x near $\vec{a}$?

$$f_1 = \text{first one}$$

$$f_2 = \text{second one}$$

$$F = (f_1, f_2)$$

$$F(\vec{a}) = (3, 6) = \vec{c}$$

$$\begin{pmatrix} \partial_2 f_1 & \partial_2 f_2 \\ \partial_3 f_1 & \partial_3 f_2 \end{pmatrix} (\vec{a}) = \begin{pmatrix} 2y & 2 \\ 2z & 3 \end{pmatrix} (1, 1, 1) = \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix}$$

This is inververtible. So by the implicit function theorem we can treat y, z as some function $h(x)$ near $1, 1, 1$ (x, y, z).

To calculate $\frac{dy}{dx}$ and $\frac{dz}{dx}$.

$$y = h(x) \text{ and } z = j(x)$$

Near the given point hte equation is going to hold,

$$x^2 + h(x)^2 + j(x)^2 = 3$$

$$x + 2h(x) + 3j(x) = 6$$

Taking a derivative,

$$2x + 2h(x)h'(x) + 2j(x)j'(x) = 0$$

$$1 + 2h'(x) + 3j'(x) = 0$$

At $(x, y, z) = (1, 1, 1)$ that becomes,

$$2 + 2h' + 2j' = 0$$

$$1 = 2h' + 3j' = 0$$

We just need to solve this system of equation for h' and j'. That same thing can be written as

$$\begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} h' \\ j' \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

This is only going to work out well if you plot it.

General Lagrange Multipliers

Constraint is given by

$$g_1(\vec{x}) = c_1$$
$$g_2(\vec{x}) = c_2$$
$$g_m(\vec{x}) = c_m$$

So $G = (g_1, \ldots, g_m)$. Here

$$G : \mathbb{R}^n \to \mathbb{R}^m$$

And let's say that X is $G^{-1}(\{\vec{c}\})$. We want to optimize $f : \mathbb{R}^n \to \mathbb{R}$ constrained to X. The idea again is that you know there are going to be certain special, f is not going to be maximized, unless something interesting happens. What's the interesting thing?

Given $\vec{a} \in X$, if f is differentiable at $\vec{a}$.

$$\{\nabla g_i(\vec{a})\} \text{ is linearly independent}$$

And $\nabla f(\vec{a})$ is not in span of $\{\nabla g_i(\vec{a})\}$. Then f is not max or min at $\vec{a}$ when restricted to X.

INtuitive Justification

Draw the picture of a sphere getting intersected by a plane. Sphere is $g_1 = c_1$ and $g_2 = c_2$ is plane. Intersection is our X.

$$X = G^{-1}(c_1, c_2)$$

Let's pick a point $\vec{a}$ right there on the intersection disk. If we are moving along the intersection then g_1, g_2 are constant. And so any tangent direction along the X are $\perp$ to $\nabla g_i(\vec{a})$. Tangent directions along $X \subset \nabla g_i^{\perp}$ $\forall i$.

$$X \subset \bigcap_{i=1}^{m} \nabla g_i^{\perp}$$

Implicit function theorem says

$$X = \cap \nabla g_i^{\perp}$$

If ∇f not in span of $\{\nabla g_i\}$ then $\exists \text{some} \vec{v} \cap \nabla g_i^{\perp}$ where $\vec{v}$ is not perp to f. Going aong that direction will increase or decrease f.

Think about ∇g_1 and ∇g_2 and they are perp to $\vec{t}$ tangent vector. ∇f is not in their span so it can't either be perp to $\vec{t}$. This diagram is necessary.

Subject ot the constraints example

$$x^2 + y^2 + z^2 = 3$$
$$x + 2y + 3z = 6$$

What is the maximum and minimum value of x?

$$f(x, y, z) = x$$

Are we guarenteed we are going to have a maximum or minimum? f is continous function. Constraint is the X which is compact.

Note Given $\vec{a} \in X$, if f is differentiable at $\vec{a}$.

$$\{\nabla g_i(\vec{a})\} \text{ is linearly independent}$$

And $\nabla f(\vec{a})$ is not in span of $\{\nabla g_i(\vec{a})\}$. Then f is not max or min at $\vec{a}$ when restricted to X.

These conditions are given in the note.

$$\nabla g_1 = (2x, 2y, 2z)$$

$$\nabla g_2 = (1, 2, 3)$$

Are they every linearly dependent? well yes but they will be linearly dependent on points that are not on X. Linearly dependent if $(x, y, z) = \kappa(1, 2, 3)$

$$\kappa^2 + (2\kappa)^2 + (3\kappa)^2 = 3$$

But we get $\kappa = \pm\sqrt{\frac{3}{14}}$. This point is outside of our required place of interest.

$$(x, y, z) = \pm\left(\sqrt{\frac{3}{14}}, 2\sqrt{\frac{3}{14}}, 3\sqrt{\frac{3}{14}}\right)$$

We don't have κ range within X (this sentence makes no sense lol). So, where is ∇f in span of $\nabla g_1, \nabla g_2$ are?

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$$

$$(1, 0, 0) = \lambda_1(2x, 2y, 2z) + \lambda_2(1, 2, 3)$$

$$\begin{aligned}
1 &= \lambda_1 2x + \lambda_2 \\
0 &= \lambda_1 2y + 2\lambda_2 \\
0 &= \lambda_1 2z + 3\lambda_2 \\
3 &= x^2 + y^2 + z^2 \qquad \text{(constraint 1)} \\
6 &= x + 2y + 3z \qquad \text{(constraint 2)}
\end{aligned}$$