Honors Multivariable Calculus : : Class 21

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$$F(x, y, z) = x^2 + y^2 + z^2$$
$$x^2 + y^2 + z^2 = 1$$
$$\vec{a} = \left(\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2} \right)$$
$$\nabla F = (2x, 2y, 2z)$$
$$\nabla F(\vec{a}) = \langle \sqrt{2}, 1, 1 \rangle$$
$$\frac{\partial f}{\partial z}(\vec{a}) \neq 0$$

Locally on the sphere z is a function of x, y

$$\frac{\partial z}{\partial x} = -\frac{\sqrt{2}}{1} = \sqrt{2}$$

We are moving but we are trying to move so that f remains the same. So we are actually trying to figure out, x_i moves by a little bit then it will change f. Then, how can x_n change so that f stays constant.

Alternatively,

$$x^2 + y^2 + h(x, y)^2 = 1$$

And then we can get with partials around x,

$$2x + 0 + 2hh_x(x, y) =$$

Let's talk about $y^3 - x^2 = 0$. Here

$$F(x, y) = y^3 - x^2$$

We want to take this near the origin (0,0). The partial derivatives are both zero at origin. Theorem does not apply. But does that y is not function of f at near $0, 0$? Actually, y is a function of x.

A justification

$$n = 3, \quad \vec{a} = (a_1, a_2, a_3)$$
$$F = F(x, y, z)$$
$$\partial f / \partial z(\vec{a}) > 0$$

Show a point $\vec{a}$ as a dot. The partial in the z direction is positive. z be along the vertical. Partial derivatives continuous. Now zoom in so that the region we are worried about, everywhere the $\partial f / \partial z$ is greater than 0. This means means that along z, F is always increasing, but we don't want to go too far. How far do we want to go? Well cut it off at some point range. Now $f(\text{upper bound}) = c + \epsilon_1$ and $f(\text{lower bound}) = c - \epsilon_2$. Because the function is continuous, so the variation of x, y is bounded by some certain m.

$$\left| \frac{\partial f}{\partial x} \right|, \left| \frac{\partial f}{\partial y} \right| < M$$

We don't want to move too much horizontally or forward-backward in a way that f doesn't change drastically. F cannot hit C twice because it's always increasing. The point on fiber is $x, y, h(x, y)$.

$$h(a_1, a_2) = a_3$$

In order to show h to be differentiable we are just going to check the partial derivatives,

$$\partial_x h(a_1, a_2) = \lim_{j \to 0} \frac{h(a_1 + j, a_2) - h(a_1, a_2)}{j}$$

We know that

$$F(a_1 + j_1, a_2, h(a_1 + j, a_2)) = c$$

similarly

$$F(a_1, a_2, a_3) = C$$

Now moving horizontally for a small for f will have to be exactly opposite small f variation and

$$\Delta F_x = -\Delta F_y$$

Because we have an equal F equipotential region.

$$\frac{\partial F}{\partial x} \cdot j = \frac{-\partial F}{\partial z} \cdot (h(a_1 + j, a_2, a_3))$$

But this upper term divided by j is exactly what we care about, the derivative of $h(x, y)$.

$$\partial_x h(a_1, a_2) = \lim_{j \to 0} \frac{h(a_1 + j, a_2) - h(a_1, a_2)}{j} = -\frac{\partial_x F(\vec{a})}{\partial_z F(\vec{a})}$$

Non baby version

Let's take $x^2 + y^2 + z^2 = 3$. This is a sphere in $\mathbb{R}^3$. But then you can also say that, $x + 2y + 3z = 6$ let's consider them both together. Intersection is going to be some curve, for this case. Kind of hard to come up with equations but near the point of intersection. Near $1, 1, 1$ can we write, we can write y, z together as an implicit function of x. Can you come up with an x so that $h(y, z) = x$.