Honors Multivariable Calculus : : Class 20

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Note The focus must be to kind of acquire the content of the class, not just blatantly take notes and shit.

The ball around $f(\vec{a})$ would have both bigger and smaller function values and that is then uninteresting.

Theorem 1. $g : \mathbb{R}^n \to \mathbb{R}$ and C^1.

$$X = g^{-1}(\{c\}), \quad \vec{a} \in X$$

If $\vec{a}$ is uninteresting, for f on X, then $\forall r > 0$, there are points on X within r from $\vec{a}$ where f is bigger and smaller than $f(\vec{a})$

Proof. Recall $p : \mathbb{R} \to X$ is any differentiable curve with point $p(t_0) = \vec{a}$ then $p'(t_0) \perp \nabla g(\vec{a})$.
Claim: Every $\vec{v} \in \nabla g(\vec{a})^{\perp}$ is $p'(t_0)$ for some p as above.
Since $\nabla f(\vec{a}) \neq \lambda \nabla g(\vec{a})$, then $\exists \vec{v} \in \nabla g(\vec{a})^{\perp}$ such that $\vec{v}$ is not perpendicular to $\nabla f(\vec{a})$.
By claim $\exists p : \mathbb{R} \to X$ with $p(t) = \vec{a}$ and $p'(t) = \vec{v}$. Consider $j : \mathbb{R} \to \mathbb{R}$. Given by $j(t) = f(p(t))$,

$$j'(t) = df_{p(t)}(p'(t))$$

$$j'(t_0) = df_{p(t_0)}(p'(t_0))$$

$$df_{\vec{a}}(\vec{v}) = \nabla f(\vec{a}) \cdot \vec{v} \neq 0$$

So j is smaller than $j(t_0)$ on one side and bigger than the other. □

Example

$$g(x, y, z) = x^2 + y^2 + z^2$$

So Let a sphere be

$$g^{-1}(\{1\}) = X$$

Let's say there is a point on X called $\vec{a} = \frac{1}{\sqrt{3}, \frac{1}{\sqrt{3}, \frac{1}{\sqrt{3}}}}$ lol silly,

$$\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$$

Attached diagram.

1 Implicit function theorem

$$x^3 + xy + e^y = 2$$

Globally y is not a function of x. y is implicitly a function of x, near $1, 0$ if things turns out nicely, we can see a little piece of the curve and looks like y is a function of x and we treat it like

$$x^3 + xh(x) + e^{h(x)} = 2$$

Then we differentiate and we treat $y = h(x)$ as an implicit function only for that point.

Theorem 2. (Baby version) $F : \mathbb{R}^n \to \mathbb{R}$ and C^1. Take a point $\vec{a}$ in $\mathbb{R}^n$ where $F(\vec{a}) = C$. We are going to suppose that $\partial F / \partial x_n$ at $\vec{a}$ is not 0. Then well

$$x_n \text{ is an implicit function of } x_1, \ldots, x_{n-1}$$

given the equation $F(x_1, \ldots, x_n) = C$. This is near $(a_1, \ldots, a_{n-1})$.

Setting $\vec{a} = (a_1, \ldots, a_n)$ we have an open $U \subset R^{n-1}$ around $\vec{a}$ and $V \subset \mathbb{R}^n$ around $\vec{a}$. And

$$h : U \to \mathbb{R}$$

Such that $V \cap F^{-1}(\{C\})$ is the graph of h. I.E there is a $(b_1, \ldots, b_n) \in V \cap F^{-1}(\{C\}) \iff$

$$b_n = h(b_1, \ldots, b_{n-1})$$