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Definition 1. Suppose $g : \mathbb{R}^n \to \mathbb{R}$ is C^1. Let

$$X = \text{ Level Surface } g^{-1}(\{c\}) = \{\vec{x} \in \mathbb{R}^n : g(\vec{x}) = \vec{c}\}$$

Take $f : \mathbb{R}^n \to \mathbb{R}$ and we say that $\vec{a} \in X$ is "interesting" for f on X if any one of the following is

- f not differentiable $\vec{a}$
- $\nabla g(\vec{a}) = \vec{0}$
- $\nabla f(\vec{a}) = \lambda \nabla g(\vec{a})$ for some scalar λ.

Theorem 1. If $\vec{a} \in X = g^{-1}(\{c\})$ is uninteresting for X, then $\forall r > 0$, $\exists \vec{b} \in X$ with $|\vec{b} - \vec{a}| < r$ and $\vec{c} \in X$ with $|\vec{b_2} - \vec{a}| < r$ such that

$$f(\vec{b_1}) > f(\vec{a}) > f(\vec{b_2})$$

on X $\vec{a}$ isn't a local min/max for f ".

Example

$$f(x) = 4x + 3y$$

On $x^2 + y^2 = 1$. Is f ever going to be non differentiable. Now what is g? It's $g(x, y) = x^2 + y^2$ and $X = g^{-1}(\{1\})$. f always being differentiable, so

$$\nabla g = \langle 2x, 2y \rangle$$

This is never $\vec{0}$ on X. So only interesting points are at

$$\nabla f = \lambda \nabla g$$

$$\langle 4, 3 \rangle = \nabla \langle 2x, 2y \rangle$$

$$4 = 2\lambda x$$

$$3 = 2\lambda y$$

And $x^2 + y^2 = 1$. Solve these for interesting points $\frac{4}{3} = \frac{x}{y}$ and

$$4y = 3x$$

$$y = \frac{3}{4}x$$

$$x^2 + \frac{9}{16}x^2 = 1$$

So we can have

$$2Sx^2 \frac{1}{16} = 1$$

$$x = \pm \frac{4}{5}$$

$$y = \pm \frac{3}{5}$$

We know f attains its min and max on X since is a cpt and f is continuous. So max at $4/5, 3/5$ and min at other.

Example

$$f(x,y,z) = x^2 - y + z$$

on X the unit sphere $x^2 + y^2 + z^2 = 1$, We call that $g(x,y,z)$. So,

$$\nabla g = \langle 2x, 2y, 2z \rangle$$

$$\nabla f = \langle 2x, -1, 1 \rangle$$

Now what we get is, and required to solve λ,

$$2x = \lambda 2x$$

Gives above either $\lambda = 1$ or $x = 0$.

$$-1 = \lambda 2y$$

$$1 = \lambda 2z$$

$$x^2 + y^2 + z^2 = 1$$

Case 01 : $x = 0$ from there

$$\frac{-1}{1} = \frac{\lambda 2y}{\lambda 2z}$$
$$z = -y$$

Using this

$$0^2 + y^2 + (-y)^2 = 1 \implies y = \pm\frac{1}{\sqrt{2}} \text{ and } z = -\left(\pm\frac{1}{\sqrt{2}}\right)$$

Case 02: $\lambda = 1$ then from there we can find

$$y = -\frac{1}{2}, z = \frac{1}{2}$$

$$x^2 + \frac{1}{4} + \frac{1}{4} = 1$$

$$x = \pm\frac{1}{\sqrt{2}}$$

2 more interesting points are

$$\left(\pm\frac{1}{\sqrt{2}}, -\frac{1}{2}, \frac{1}{2}\right)$$

X is compact then we will get min max.