Honors Multivariable Calculus : : Class 17

February 16, 2024

Ahmed Saad Sabit, Rice University

Definition 1. Proposition: If f is differentiable at $\vec{a}$ ut $\mathrm{d}\vec{f_a}$ is not not identically zero, then $\vec{a}$ is not a local extremum for $\vec{f}$.

Consequently if f has a local maximum at $\vec{a}$, then $\vec{a}$ is a critical point for f. ($\mathrm{d}f_{\vec{a}} = 0$ for f is differentiable at $\vec{a}$.)

Proof. Suppose $\mathrm{d}f_{\vec{a}}$ is not identially 0. Then $\exists \vec{v} \in \mathbb{R}^n$ such that $\mathrm{d}f_{\vec{a}}(\vec{v}) \neq 0$.

$$\lim_{t \to 0} \frac{f(\vec{a} + t\vec{v}) - f(\vec{a})}{t} = D_{\vec{v}} f(\vec{a})$$
$$= g'(0)$$

where $g(t) = f(\vec{a} + t\vec{v})$. So $t = 0$ is a not a locla minimum or maximum for g.
So $\forall \varepsilon > 0$ let $\varepsilon' = \frac{\varepsilon}{|\vec{v}|}$ We know that $\exists t_1, t_2 \in (-\varepsilon', \varepsilon')$ such that

$$g(t_1) > (g(0)) > g(t_2)$$

Since 0 is not a local min or max for g so

$$f(\vec{a} + t_1\vec{v}) > f(\vec{a}) > f(\vec{a} + t_2\vec{v})$$

Now here,

$$\vec{a} + t_1\vec{v}$$

and

$$\vec{a} + t_2\vec{v}$$

are within the ε of $\vec{a}$. and f is bigger than that of $f(\vec{a})$ at one of them. Smaller than $f(\vec{a})$ at the other. So $\vec{a}$ is not a local extremum. □

Definition 2. Prop: Suppose $f : \mathbb{R}^n \to \mathbb{R}$ and $\mathrm{d}f_{\vec{a}} = 0$. Then consider the Quadratic Form

$$Q(\vec{x}) = \vec{x}^t H \vec{x}$$

Where H is the Hessian Matrix at $\vec{a}$.

- If Q is positive definite, then f has a local min at $\vec{a}$.
- If Q is negative definite, then f has a local max at $\vec{a}$.
- If Q is indefinite, then f has neither a local max or min at $\vec{a}$.

Proof. Suppose $\mathrm{d}f_{\vec{a}} = 0$ and H is positive. We know that $f(\vec{a} + \vec{h})$ is $f(\vec{a}) + \mathrm{d}f_{\vec{a}}(\vec{h}) + \frac{1}{2!}\vec{h}^t H \vec{n} + R_2(\vec{a}, \vec{h})$

$$f(\vec{a} + \vec{h}) - f(\vec{a}) = \frac{1}{2!}\vec{h}^t H \vec{h} + R_2(\vec{a}, \vec{h})$$

So the idea is that the difference is basically the left side term in the right of equality. The remainder has to be quite small. So the intuitive idea is a domination. Suppose all eigen values of H are positive.

$$H = PDP^t$$

P orthogonal, Define $\vec{j}$ such that

$$\vec{j} = P^t \vec{h}$$

And

$$= \frac{1}{2!} \vec{j}^t D \vec{j}$$

$$h^t H h = h^t P D P^t h$$

Where $h^t P = j^t$ and $P^t h = j$.

$$= \frac{1}{2!} j^t D j + R_2(\vec{a}, \vec{h})$$

$$= \frac{1}{2!} \left(\lambda_1 j_1^2 + \lambda_2 j_2^2 + \ldots + \lambda_n j_n^2 \right) + R_2(\vec{a}, \vec{h})$$

$$\geq \frac{1}{2!} (\lambda_n j_1^2 + \ldots + \lambda_n j_n^2) + R_2(\vec{a}, \vec{h})$$

$$= \frac{1}{2!} \lambda_n^2 |\vec{j}|^2 + R_2(\vec{a}, \vec{h})$$

$$= \frac{\lambda_n}{2!} |h|^2 + R_2(\vec{a}, \vec{h})$$

But we know that

$$|R_2(\vec{a}, \vec{h})| < \frac{\lambda_n}{2!} |h|^2$$

Because

$$\lim_{h \to 0} \frac{R_2}{h^2} = 0$$

And R_2 is strictly > 0. □