Honors Multivariable Calculus : : Class 16

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Quadratic Forms

A quadratic forms is a function of n real variables. Such that

$$f : \mathbb{R}^n \to \mathbb{R}$$

of the form $Q(x_1, \ldots, x_n) =$ a polynomial in $x_1, \ldots, x_n$ here all the terms are of degree 2.

$$Q(x_1, x_2, x_3) = x_1^2 + 4x_1x_2 + x_2^2 + 10x_1x_3 + 2x_3^2$$

Definition 1. A quadratic form Q is positively defined $Q(\vec{x}) > 0$ always being positive definite. Positive Semi-Definite $Q(\vec{x}) \geq 0$, negative definite $Q(\vec{x}) < 0$ and $Q(\vec{x}) \leq 0$. Indefinite if Q is sometimes > 0 and sometimes < 0. For some quadratic forms it's easy to tell.

Examples of Quadratic Forms

We know the positive definite is always

$$Q(x_1, x_2, x_3) = x_1^2 + 4x_2^2 + 7x_3^2$$

Definition 2. Fact: For every Quadratic form $Q : \mathbb{R}^n \to \mathbb{R}$, there is a symmetric matrix A such taht

$$Q(\vec{x}) = \vec{x}^{+} A\vec{x}$$

$$Q(\vec{x}) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 5 \\ 2 & 1 & 1/2 \\ 5 & 1/2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Any quadratic form can be turned into this matrix format. The above one can yield us

$$Q(x_1, x_2, x_3) = x_1^2 + 4x_1x_2 + x_2^2 + 10x_1x_3 + 2x_3^2$$

SO

$$Q(\vec{x}) = \vec{x}^t PDP^t \vec{x}$$

Let $\vec{y} = P^t\vec{x}$ and hence we have D to have the eigenvalues of A. You take Eigenvalues and use the spectral theorem. Look at the signs and that tells you the answer.

Derivative Zeroes

Definition 3. Let $f : D \to \mathbb{R}$ such that $D \subset \mathbb{R}^n$ and $\vec{a}$ is interior to D. We say that f has a local max at $\vec{a}$ if $\exists r > 0$ such that $\forall \vec{x} \in B_r(\vec{a})$ that $f(\vec{a}) \geq f(\vec{x})$. For minimum just flip sign.

It's basically single variable calculus just using the definition of a ball.

Definition 4. Proposition: If $f : \mathbb{R} \to \mathbb{R}$ has a local minimum or maximum at a then either $f'(a)$ is zero, or f is not differentiable at a.

I don't have time to prove this, but I will state it

Theorem 1. If f has a local extremum at $\vec{a}$ then either f is not differentiable $\vec{a}$ or $df_a = 0$.