Honors Multivariable Calculus : : Class 15

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Taylor's Expansion for Multivarible Functions

1-variable 2nd order Taylor Polynomial $x = a$ is

$$f(a) = f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \cdots$$

This matches the 0th and 1-st derivative of f at a. If $x \approx a$

$$f(a+h) \approx f(a) + f'(a)h + \frac{1}{2!}f''(a)h^2$$

When $h \approx 0$.

Now we are going to do this for multivariable function,

There are taylor functions that converge but not the function it's expanding.

Two Variable function

$$f(0,0) + (\text{something})x + (\text{something})y$$

$$f(0,0) + \frac{\partial f}{\partial x}(\vec{0})x + \frac{\partial f}{\partial y}(\vec{0})y$$

But we should have x^2, y^2, xy terms

$$\text{incoorect!} = f(0,0) + \frac{\partial f}{\partial x}(\vec{0})x + \frac{\partial f}{\partial y}(\vec{0})y + x^2 + y^2 + xy$$

$$f(0,0) + \frac{\partial f}{\partial x}(\vec{0})x + \frac{\partial f}{\partial y}(\vec{0})y + \frac{1}{2!}\frac{\partial^2 f}{\partial x^2}f(\vec{0})x^2 + \frac{1}{2!}\frac{\partial^2 f}{\partial y^2}(\vec{0})y^2 + \frac{\partial^2 f}{\partial x \partial y}(\vec{0})xy$$

$$f(0,0) + \frac{\partial f}{\partial x}(\vec{0})x + \frac{\partial f}{\partial y}(\vec{0})y + \frac{1}{2!}\frac{\partial^2 f}{\partial x^2}f(\vec{0})x^2 + \frac{1}{2!}\frac{\partial^2 f}{\partial y^2}(\vec{0})y^2 + \frac{1}{2!}\frac{\partial^2 f}{\partial x \partial y}(\vec{0})xy + \frac{1}{2!}\frac{\partial^2 f}{\partial y \partial x}(\vec{0})yx$$

What about even more terms? We will have additional

$$\frac{1}{3!}\frac{\partial^3 f}{\partial x^3}x^3 + \frac{1}{3!}\frac{\partial^3 f}{\partial y^3}y^3 + \frac{1}{2}\frac{\partial^3 f}{\partial x^2 \partial y}x^2 y + \frac{1}{2}\frac{\partial^3 f}{\partial y^2 \partial x}y^2 x$$

Think about this

$$\frac{1}{3!}(xxy + xyx + yxx)$$

So the k'th order terms for the taylor polynomials for some function $f : \mathbb{R}^n \to \mathbb{R}^m$ are (an outline first, exact one later)

$$\sum_{i_k \in \{1 \dots k\}} \frac{\partial^k f}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_k}} (x_{i_1} x_{i_2} \cdots x_{i_k})$$

If it is centered at $\vec{a}$ then $\vec{a} = (a_1, a_2, \dots, a_n)$ then replace $\vec{x}_i$ with $\vec{x}_i - \vec{a}_i$

How correct is our taylor series?

Single variable

$$f(a+h) = f(a) + f'(a)h + \ldots + \frac{1}{k!}f^{(k)}(a)h^k + R_k(a,h)$$

Here $R_k(a,h)$ is the remainder of k order.

Facts about R_k: If f is C^k near a then

$$\lim_{h\to 0} \frac{R_k(a,h)}{h^k} = 0$$

If f is C^{k+1} then

$$\begin{aligned} R_k(a,h) &= \int_a^{a+h} \frac{(a+h-x)^k}{k!} f^{(k+1)}(x)\mathrm{d}x \\ &= \frac{1}{(k+1)!} f^{k+1}(c)h^{k+1} \end{aligned}$$

Let's consider the first orders for here with $\vec{a} = (a_1, a_2)$ and $\vec{h} = (h_1, h_2)$

$$f : \mathbb{R}^n \to \mathbb{R}$$

We can't call it equality unless we have added $R(\vec{a}, \vec{h})$

$$f(\vec{a}+\vec{h}) = f(\vec{a}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{a})h_i + R(\vec{a}, \vec{h})$$

$$f(\vec{a}+\vec{h}) = f(\vec{a}) + df_{\vec{a}}(\vec{h}) + R_1(\vec{a}, \vec{h})$$

But from the raw definition of the derivative $df_{\vec{a}}(\vec{h})$ setting $\vec{h} \to 0$ we can get

$$\frac{\left|R_1(\vec{a}, \vec{h})\right|}{\left|\vec{h}\right|}$$

This will go to zero. For the second order we want,

$$f(\vec{a}+\vec{h}) = f(\vec{a}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{a})h_i + \frac{1}{2!}\sum_{i_1, i_2=1}^{n} \frac{\partial^2 f}{\partial x_{i_1}\partial x_{i_2}}(\vec{a})h_{i_1}h_{i_2} + R_2(\vec{a}, \vec{h})$$

The remainder here

$$\frac{\left|R_2(\vec{a}, \vec{h})\right|}{\left|\vec{h}\right|^2}$$

This will go even faster than zero. If f is C^3 near $\vec{a}$ then $R_2(\vec{a}, \vec{h})$ is

$$\frac{1}{3!}\sum_{i,j,k=1}^{n} \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}(c_{ijk})h_i h_j h_k$$

The idea to prove this is to use the one single variable calculus remainders on $g(t) = f(\vec{a} + \vec{h}t)$.

$$= \frac{1}{2!}\left(h_1, \ldots, h_n\right) \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(\vec{a}) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(\vec{a}) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1}(\vec{a}) \end{pmatrix} \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}$$