Honors Multivariable Calculus : : Class 14

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For $f : \mathbb{R}^n \to \mathbb{R}$ the first order derivatives are

$$\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}$$

There are are all $\mathbb{R}^n$ to $\mathbb{R}$.

Differentiate ∂f_{x_i} with respect to x_j, so

$$\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)$$

In more condensed form

$$\frac{\partial^2 f}{\partial x_j \partial x_i} = (f_{x_i})_{x_j}$$

An example can be

$$f(x,y) = x\sin(y-x) + y\sin(x+y) - y$$
$$f_x = \sin(y-x) + x\cos(y-x)(-1) + y\cos(x+y)\cdot 1$$
$$f_y = x\cos(y-x) + \sin(x+y) + y\cos(x+y)\cdot 1 - 1$$
$$f_{xx} = \cos(y-x)\,(-1) - \cos(y-x) + x\sin(y-x)(-1) - y\sin(x+y)$$
$$f_{yy} = -x\sin(y-x) + \cos(x+y) + \cos(x+y) - y\sin(x+y)$$
$$f_{xy} = \cos(y-x) + x\sin(y-x) + \cos(x+y) - y\sin(x+y)$$
$$f_{yx} = \cos(y-x) + x\sin(y-x) + \cos(x+y) - y\sin(x+y)$$

Clairaut's Theorem

Theorem 1. If f is C^2, not only the partial derivatives exist but also the function is continuous. So at some point $\vec{a}$ then mixed partials are equal.

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$$

Proof. For simplicity write f as $f(x,y)$. Let

$$\vec{a} = (x_0, y_0)$$

Define

$$S(\Delta x, \Delta y) = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0 + \Delta x, y_0) - (f(x_0, y_0 + \Delta y) - f(x_0, y_0))$$

Define another function $g(x)$

$$g(x) = f(x, y_0 + \Delta y) - f(x, y_0)$$

Then

$$S(\Delta x, \Delta y) = g(x_0 + \Delta x) - g(x_0)$$

Using the mean value theorem on G because it is a single variable function, and g is also differentiable because f is differentiable and we have single order and this is going to be

$$= \Delta x g'(c) \text{ for some } c \text{ between } x_0 \text{ and } x_0 + \Delta x$$

$= \Delta x \left(f_x(c, y_0 + \Delta y) - f_x(c, y_0) \right) = \Delta x \left(\Delta y f_{xy}(c, d) \right)$ for some d between y_0 and $y_0 + \Delta y$ as Mean Value Theorem

So

$$\frac{S(\Delta x, \Delta y)}{\Delta x \Delta y} = f_{xy}(c, d)$$

Here c, d is some coordinate in the rectangle. But as $\Delta x, \Delta y \to 0$ we need $c, d \to 0$ and hence the position become $\vec{a} = (x, y)$. Using continuity

$$f_{xy}(c, d) \to f_{xy}(\vec{a})$$

This is the limit as $\Delta x, \Delta y \to 0$ as $\lim_{\Delta x, \Delta y \to 0} S(\Delta x, \Delta y)/\Delta x \Delta y = f_{xy}(\vec{a})$. We can do the exact same process in the opposite order we can simply form the f_{yx} □

Partial differentiation draw a point $\vec{a}$ and extend it two both directions by Δx and Δy, so you have a rectangle of Δx and Δy dimensions.

Taylor Polynomial

Whats the second order Taylor Polynomial for f at $x = a$?

$$f(x) \approx f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2$$

When x is near a. Why is the 2! there? Because we want the second derivative to match the left side of the equation.